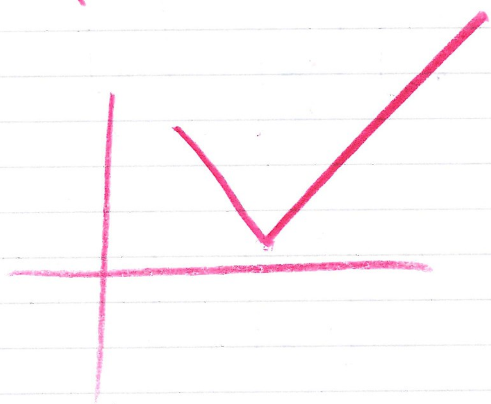
10 ① S

$\{1, 2, 3\}$

$1+2 \neq 2+1 \Rightarrow$ Hazard

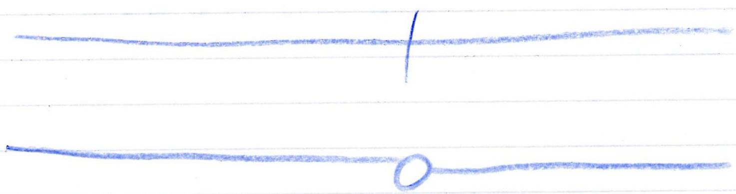


Set P.12-13 of L Note

② $T : U \to V$

③ $\mathbb{R}^2_{\geq 0}$ ✗ $\qquad \mathbb{R}^2$ ✓

④



$\mathbb{R}$

$\mathbb{R} \setminus \{2\} = M$

2

A1 ✓

A3 ✓

A2 ✓

A4 : $\forall a \in M, \exists -a \cdots$

$\exists a \in M$, s.t. $\nexists -a$ negation

let $a = -2 \cdots$ $\square$

P2

Ex Sheet 4
Q 2.1

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \cancel{x_m}\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$n=3$, $m=2$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

Statement I Given $A\vec{x} = \vec{b}$ exists

Statement II — $A\vec{x} = \vec{b}$ has a solution

$$\left\{ \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \right\}.$$

$\exists\ c_1, \ldots, c_n \in \mathbb{R}$ st.

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

P3

~~Span~~ Logic

A if B	$B \Rightarrow A$
A if and only if B	$A \Leftrightarrow B$
A only if B	$B \Leftarrow A$
if A, then B	$A \Rightarrow B$
$A \Rightarrow B$	$(\neg B) \Rightarrow (\neg A)$

Linear dependent

$\{a_1, \cdots, a_{n+1}\}$ e.g. In $\mathbb{R}^2$ $(1,2)$ $(2,4)$

$\boxed{(1,2) \quad (0,0)}$

Given any $c_i \in \mathbb{R}_d$, where c may not be ~~zero~~,

$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_{n+1}\vec{a}_{n+1} = \vec{0}$

$\dfrac{a_n}{a_{n-1}}$ exists $\Leftrightarrow$ $a_{n-1} \neq 0$

$\exists$ $c_1, \cdots, c_n \in \mathbb{R}$ s.t.

$a_{n+1} = c_1 a_1 + \cdots + c_n a_n$

e.g. $n=1$ let $a_2 = (0,0)$, $a_1 = (5,6)$.

let $C_1 = 0$

see P5

P4

St' I $\Rightarrow$ St' II.

St' I

~~$a_{11} + a_{12} + \dots + a_{1n} = b$~~

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right]$$

← Conjecture

~~$\Rightarrow c_1 = x_1, \ x_2 \ c_2 =$~~

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

D

Main part Done.

Conflict with something?

II $\Rightarrow$ I.

satisfy conflict

P5

Conjecture: $\left\{ \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ is linearly dependent.

Proof:

$$c_1 \begin{pmatrix} 5 \\ 6 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ if linearly dependent.

$$5c_1 = 0 \quad \Rightarrow c_1 = 0.$$
$$6c_1 = 0$$

$$c_2 \in \mathbb{R}$$

$\square$

$n = 1 \quad m = 2$

$$\left\{ \begin{pmatrix} a_{11} \\ \vdots \\ a_{21} \end{pmatrix}, \cancel{\begin{pmatrix} a_{12} \\ \vdots \end{pmatrix}}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} \right\} \quad \exists \; c_1 \in \mathbb{R} \text{ s.t.}$$

St I

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$ set $b_1 = 0, \; b_2 = 0$

& $a_{11} = 5, \; a_{21} = 6.$

$$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ 6 \end{pmatrix} \quad \therefore c_1 = 0$$

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix} x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightarrow x_1 = 0$

p.6

WARM UP 5:

$\mathrm{Span}\langle a, b\rangle = \{\lambda_1 a + \lambda_2 b \mid \lambda_1, \lambda_2 \in \mathbb{K}\}$.

normally, $\mathbb{K} = \mathbb{R}$.

~~$\mathrm{span}\{\vec{a}, \vec{b}, \vec{c}\}$~~

$\mathrm{span}\{\vec{a}, \vec{b}, \vec{c}\} = \{\lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 \vec{c} \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}\}$

$\{a, b\}$ spans P

if $P = \mathrm{span}\langle a, b\rangle$

$A \in \mathbb{R}^{2 \times 2}$

$$A = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} r_1 \\ \hline r_2 \end{bmatrix} = \left[v_1 \mid v_2 \right]$$

P7

Ex. 5

2.(1) $\mathrm{I} \Longrightarrow \mathrm{II}$

$\{\vec{v}_{c(1)}, \dots, \vec{v}_{c(m)}\}$ is a basis for K^m

$\tilde{A} = \left(\vec{v}_{c(1)} \middle| \vec{v}_2 \middle| \vec{v}_{c(2)} \middle| \cdots \middle| \vec{v}_n\right)$

~~span$\{\vec{v}_1, \dots, \vec{v}_n\}$~~

$= \{$

$\{\vec{v}_1, \dots, \vec{v}_n\}$ spans K^n

~~$A = \{\vec{v}_1, \dots$~~

$A = \left(\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_n\right)$

Write the r.r. form of A as $\tilde{A}$.

$$\tilde{A} = \begin{array}{c} \begin{pmatrix} * & & & \\ 0 & * & * & \\ 0 & & & \ddots \\ \vdots & & & * \\ 0 & \cdots & \cdots & 0 \end{pmatrix} m \\ n \end{array}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

P8

$$\begin{pmatrix} 1 & 10 & 2 & 5 & 7 \\ 0 & 0 & 1 & 0 & 3 \\ & & & 1 & \pi \\ & & & & 1 \end{pmatrix} \in \mathbb{R}^{4\times 5}$$

$\cancel{5\times 6}$

$\cancel{6\times 7}$

i	1	2	3	4	~~5~~	~~6~~
$c(i)$	1	3	4	5	~~7~~	

$\vec{v}_{c(1)}, \vec{v}_{c(2)}, \vec{v}_{c(\ldots}$

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \cancel{\begin{pmatrix}10\\0\\0\\0\\0\end{pmatrix}}, \cancel{\begin{pmatrix}5\\0\\1\\0\\0\end{pmatrix}}, \cancel{\begin{pmatrix}7\\3\\ \\ \\ \end{pmatrix}} \begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}5\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}7\\3\\\pi\\1\\0\end{pmatrix}$$

By shifting, $\mathcal{B}$ is a basis.

P9

II equivalent to:

$\{\vec{v}_{c(1)}, \ldots, \vec{v}_{c(m)}\}$ is a basis for K^m, where

~~$\alpha \vec{v}_{c(i)}$~~ i) $\vec{v}_{c(i)}$

$\alpha_1 \vec{v}_{c(1)} + \alpha_2 \vec{v}_{c(2)} + \cdots + \alpha_m \vec{v}_{c(m)} = \vec{0}$ ~~where~~ $\Longleftrightarrow$

$\alpha_i = 0$ for linear independence.

ii) $\forall\ x \in K^m$, ~~span~~ ~~$\langle$~~ ~~$\exists$~~

~~span~~ $\exists\ \lambda_i \in \mathbb{R}$ s.t

$x = \lambda_1 \vec{v}_{c(1)} + \ldots \lambda_m \vec{v}_{c(m)}$

~~span $\langle \vec{v}_1, \ldots$~~

$\text{span} \langle \vec{v}_{c(1)}, \ldots, \vec{v}_{c(m)} \rangle = \{ \lambda_1 \vec{v}_{c(1)} + \cdots + \lambda_m \vec{v}_{c(m)} \mid \lambda_1, \ldots \lambda_m \in \mathbb{R} \}$

P10

Conjecture: $\forall i$, then

$$\overrightarrow{V_{C(i)}} = \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then $\{\overrightarrow{V_{C(1)}}, \dots, \overrightarrow{V_{C(m)}}\}$ spans K^m.

Pick any $\vec{x} \in K^m$,

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_m \end{pmatrix}$$

Consider $\vec{x} = \lambda_1 \overrightarrow{V_{C(1)}} + \dots + \lambda_m \overrightarrow{V_{C(m)}}$

where $\lambda_m = x_m$

$\lambda_{m-1} = x_{m-1} + x_m \cancel{a} \left(\overrightarrow{V_{C(m)}}\right)_{m-1}$

$$\begin{pmatrix} \vdots \\ x_{m-1} \\ x_m \end{pmatrix} = \lambda_1 \begin{pmatrix} \vdots \\ 0 \\ 0 \end{pmatrix} + \dots + \lambda_{m-1} \begin{pmatrix} \vdots \\ 1 \\ 0 \end{pmatrix} + \lambda_m \begin{pmatrix} \vdots \\ a_{m-1} \\ 1 \end{pmatrix}$$

Suppose

$$\lambda_1 \overrightarrow{v_{c(1)}} + \cdots + \lambda_m \overrightarrow{v_{c(m)}} = 0.$$

$$\lambda_1 \begin{pmatrix} \vdots \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_{m-1} \begin{pmatrix} \vdots \\ \vdots \\ 0 \end{pmatrix} + \lambda_m \begin{pmatrix} \vdots \\ \vdots \\ 1 \end{pmatrix} = 0.$$

$$\lambda_m = 0.$$
$$\lambda_{m-1} = 0.$$

$\square$