Does the negative interest rate decrease consumption and investment in Japan? — An insight from a Bank-Credit Model

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ABSTRACT

This paper builds a micro-founded macroeconomic model and finds an uncertain answer to whether negative interest rate decreases consumption and investment in Japan. The insufficient studies on Japan and the current debates on negative interest rate motivate me to study this topic. Empirical data with key literature are presented to build and solve my model. An intuitive micro-founded model is constructed, with reference to key observations found in Japanese data and literature. Theoretical mathematics is utilised to solve the model, followed by numerical results which suggest an uncertain answer to the topic question. Infinitely many possibilities are found which support commercial banks to cease lendings after negative interest rate being adopted, thus resulting in a decreased consumption and investment. However, there are also infinitely many possibilities being found that supports an unchanged consumption and investment. Lastly, extensions are drawn for the model, followed by an evaluation.

Keywords: Negative Interest Rate, Microfoundations, Macroeconomic Modelling, Mathematical Methods.
JEL Classification: E50, E44, C61, C65.

Note: The first edition of this paper was submitted to the Department of Economics, University of Warwick, in partial fulfilment of the requirements of the degree of Bachelor of Science (with Honours) in Mathematics and Economics.

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To my mum Xiaofei Yang,

who encouraged me to study, educated me,

and funded me throughout my period of undergraduate study.

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SUPERVISOR'S FEEDBACK ON FIRST EDITION

The attempt to build a micro-founded macro model from scratch is well beyond a standard undergraduate project. You successfully do this and the model introduces novel features in the behaviour of commercial banks which describe basic features of the Japanese economy and simultaneously make the model very difficult to solve. The analytical insights on display to come up with the existence theorem and recipe for maximisation are impressive. The division of the decision space of 'farmers' into the three subsets is both intuitive — in that each set has a behavioural description — and a powerful tool for breaking the problem into solvable parts. While the maths set up is very heavy even compared to modern macro models, the payoff is that you are able to analyse a situation in which the non-differentiability renders typical econ results inapplicable. The key insight seems to be that the response of an economy to a negative interest rate is extremely sensitive to the fine detail of the assumptions made about functional form and parameter values, even in the highly stylised environment you present. You successfully show reductions in consumption and output could follow from the central bank adopting a negative interest rate when commercial banks face the constraints you model. One way to read the ambiguity of your main result is as a microfoundation for the total lack of agreement amongst macroeconomists as to whether negative rates are a good idea! This seems a pretty decent conclusion for such an ambitious project. On the downside, the topic is perhaps too big for an undergraduate project, so that while the model shows promise, it would probably not be able to influence the policy debate in its current form.

List of Common Abbreviations

Abbreviation	**Full Name**
BoJ	Bank of Japan
ComB	Commercial Bank
DSGE	Dynamic Stochastic General Equilibrium (Model)
e.g.	exempli gratia (Latin)
GDP	Gross Domestic Product
i.e.	id est (Latin)
int.pt.	intersection point(s) (or point(s) of intersection)
i_b	interest rate for borrowing
i_c	Central Bank interest rate
i_d	interest rate for deposit
JGB	Japanese Government Bond
MATLAB 3rd approx. or $\approx$	Approximated to the 3^{rd} decimal place by MATLAB
MATLAB 4th approx.	Approximated to the 4^{th} decimal place by MATLAB
maths	mathematics
n.a.	not available
NIR	Negative Interest Rate
UMP	Utility Maximisation Problem(s)
VaR	Value-at-Risk (Model)
w.l.o.g.	without loss of generality
ZLB	Zero Lower Bound

Note: For most of the abbreviations, when they are first mentioned, they would come within brackets after their full names.

Contents

1 Preface to second edition

One may have realised, there is no Preface to the first edition. That's an important point to make — this work is hardly an academic publication, rather it is my undergraduate dissertation with requirements on formatting and contents. For the originality, I did not remove or change any content in the sections following. I was unsure whether to publish, i.e. make the second edition available to the public through my personal website. Despite a high mark (82%, one of the highest marks amongst the cohort) was received, critiques on the work from some departmental staff came by, as I communicated with them after my graduation. After a throughout thought, I made the decision, which is to share my original work to the public via internet.

There are some changes in Japan after my first edition, the noticeable one is BoJ's hawkish decision on the yield curve control in July 2018 — the 10 year JGB yield has been lifted up by that, and this seems to offer the society a view that BoJ may stop the NIR policy in a foreseeable future. Nonetheless, NIR is still an interesting monetary policy to be discussed, in particular, whether it can be in the toolkit for central bankers in case next global recession happens, with nominal interest rate stuck at zero per cent.

As a general guidance to the readers, people with no interest in mathematical or theoretical economic content shall skip section 5, section 7, section 9, and section 10; also read with skipping throughout section 6. People with no interest in Japan-specific topics shall skip section 3 and section 4. Mathematicians who are not interested in social science shall skip section 3, section 4, section 5, and section 10.

Finally, as a personal note, I doubt if my work is worth to be extended, but I am quite sure that the monetary economists at least, will still be interested in NIR.

Parley Ruogu Yang
Cambridge
August 2018

Part I

Motivation and Literature Review

Overview: section 2 briefly introduces the topic, followed by section 3 which provides observations for the historical and contemporary Japanese monetary issues. These motivate me to build a Japan-specific model. To do so, it is crucial to make further observations in contemporary Japan, thus section 4 utilises data on interest rates, deposits, and lendings in Japan to reach the assumptions and the conjecture in my model. Then, section 5 reviews key papers in macroeconomics, microeconomics, and mathematics that are helpful to the model building and solving.

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2 Introduction

On 29th January 2016, Bank of Japan (BoJ) decided to implement a Negative Interest Rate (NIR) policy, which is to apply an interest rate of -0.1% to the excess deposits that financial institutions hold at BoJ.[1] The current governor of BoJ — Haruhiko Kuroda, who intends to pursue further monetary expansions[2], has recently been renominated to serve for another five years.[3]

It is thus critical for my research to have discussions on NIR before advising BoJ to cut their interest rate further. In particular, if NIR would cause a decrease in consumption and investment, then I would suggest BoJ avoid cutting the rate into the further negative area.

I aim to build a model with Japanese empirical elements, which admits the commercial bank's credit issuing decisions, and extend the model to deduce macroeconomic changes in consumption and investment due to NIR.

[1] Policy statement: Bank of Japan (2016).

[2] For example, in the press conference (Bank of Japan and Kuroda 2016, p.9), Kuroda mentioned "Going forward, if judged necessary, it is possible to further cut the interest rate from the current level of minus 0.1 percent".

[3] Press Source: Harding (2018). Official Announcement: Bank of Japan (2018a).

3 Motivation with Generic Literature Review

3.1 Why Japan?

> " Why should we study Japan? The world's third-largest economy plays a key role in the rising Asia story ... Japan remains, however, poorly understood ... "
>
> — Kingston (2014, p.1)

The above quote partly explains my motivation to study further on Japan. During the study, a book from Mosk (2008) regarding Japan's economic development between 1600 and 2000 gave an argument, about why we shall not fit Japanese stories in the mainstream neo-classical economics:

> " As appealing as this logic [which fits Japanese post-war miracle growth with Swan-Solow model] is, it is hard to abandon the notion that some Japan specific factors are important. "
>
> — Mosk (2008, p.284)

In my opinion, major banking and monetary models are designed to explain Western economic phenomena. Disagreements arise when comparing Japanese empirical facts against key assumptions in major Western banking and monetary models.

For example, some Swiss banks impose NIR to depositors after Swiss National Bank's NIR policy,[4] meaning that Western monetary models can assume commercial banks being able to transfer the NIR from the central bank to households. By contrast, no Japanese bank imposes NIR to their depositors,[5] thus disagree with assumptions in some monetary models.[6] Hence, those models might not fit in the Japanese context.

The above observations not only prompt me to centre my research at Japan, but also motivate me to reflect whether I shall fully adopt existing world-renowned monetary models, or build a model myself, to analyse NIR in Japan.

3.2 Why Negative Interest Rate?

Keynes (1936, pp.165-174, 194-209) and Hicks (1937) suggested that rate cut may boost consumption and investment, with a remark from Hicks (1937, pp.153-154) that interest rates must always be positive. Current macroeconomic textbooks [7] call such a remark ZLB.

An intuition is, when interest rate decreases, return on savings drops, so consumers are discouraged to save and encouraged to spend and borrow, thus consumption increases. Similarly, firms and investors face

[4] See Alternative Bank Schweiz (2018) and UBS AG (2018) for negative deposit rate examples from Swiss banks, and see Blackstone (2017) and finews.com (2017) for media sources.

[5] See rates from each Japanese commercial bank for further details, e.g. MUFG (2018), and see subsection 4.1 for overall data.

[6] More of these are explicitly shown in subsection 5.2.

[7] e.g. Blanchard (2017, p.27).

a cheaper cost of borrowing, thus more investment occurs. But if the interest rate goes below zero, then people simply cash out from their bank accounts, leaving no results for further rate cuts.

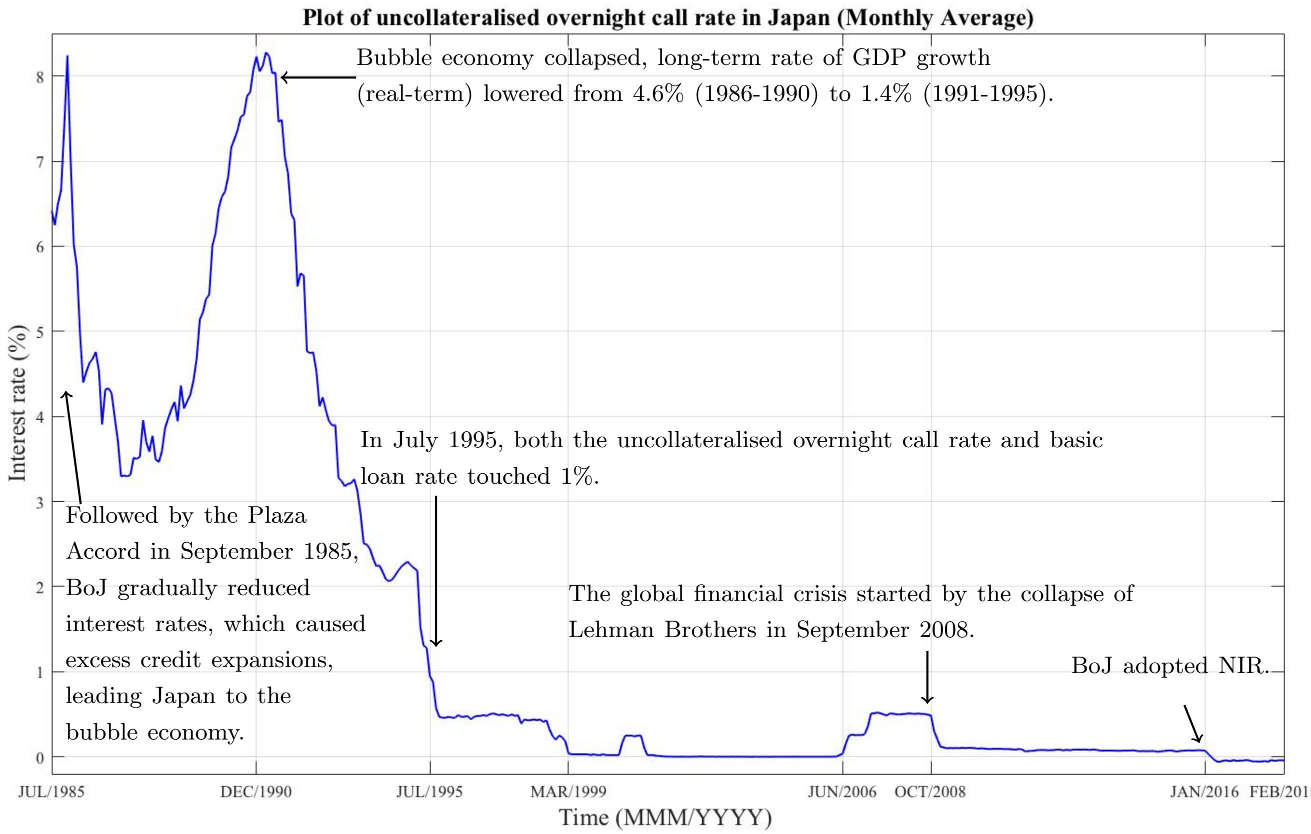

Figure 1: Short-term interest rate in Japan generally decreased since the early 1990s, then stayed between 0% and 1% during 1996-2015.
Data Source: Bank of Japan (2018b).
Text Source: Saito (2000, pp.241-246), Rossi and Malavasi (2016, pp.3-17), and Horiuchi and Otaki (2017, pp.33, 54-63).
Note on definition: Bank of Japan (2018c) gave further statistical details on the uncollateralised overnight call rate. It is also common to use the uncollateralised overnight call rate as a definition of "short-term interest rate" in Japan, and the call rate acts as a good proxy for the BoJ policy rates, even after 2016. The same approach was given by Yoshino and Taghizadeh-Hesary (2017, pp.157-158) .

Figure 1 plots the interest rate in Japan in the past three decades. In 1991, the bubble economy collapsed, and economic stagnation set in for an entire decade, despite dramatic rate cuts (Iyoda 2010, pp.74-93). As the interest rate hit 1% in 1995, ZLB issue arose. Saito (2000, p.245) narrated it as "lowering interest rates further was a difficult task for the monetary authority". Svensson (2006) summarised the Japanese monetary policies in 1995-2005 as a combination of low interest rates, zero interest rates, and a quantitative easing.[8] Despite a short recovery in 2006-2008, the interest rate was cut back to near-zero again after the 2008 financial crisis (Rossi and Malavasi 2016, pp.3-17).

[8] A similar description was given by Ito, Patrick, and Weinstein (2005, pp.107-143).

Similar to Saito (2000), Svensson (2006, pp.1,8) also believed that Japan faced ZLB in the late-1990s, which prevented BoJ from setting its optimal interest rate. These thoughts motivate policymakers to think that NIR being desired by the economy.

A more theoretical approach from Goodfriend (2016) further called for NIR:

> " Removing the zero interest bound is nothing more than the sensible application of monetary economics, ..., temporarily negative nominal interest rate policy actions are called for against deflationary recession."
>
> — Goodfriend (2016, p.29)

There are abundant arguments against NIR. One of it comes from Keynes (1936):

> " [If the interest rate is lower than a certain level (which Hicks (1937, pp.153-154) clarified as near-zero),] ... everyone prefers cash to holding a debt which yields so low interest rate. In this event the monetary authority would have lost effective control over the interest rate. "
>
> — Keynes (1936, p.207)

A modern approach to describe the above concept was given by Schmitt-Grohe and Uribe (2009, pp.89-92):

$$\begin{aligned} \lim_{M\to\infty} \frac{\frac{\partial u(c,M)}{\partial M}}{\frac{\partial u(c,M)}{\partial c}} &= 0 \quad \forall c > 0 \\ \text{where } c &\text{ is consumption,} \\ u &\text{ is the household utility function,} \\ M &\text{ is the nominal money balances.} \end{aligned} \tag{1}$$

Equation 1 means, as M becomes large enough, the marginal increase in utility of having one more unit of money is negligible compared to the marginal increases in utility for a unit increase in consumption. Schmitt-Grohe and Uribe (2009, p.89) then induced "money demand approaches infinity as the nominal interest rate vanishes". This shows that NIR is never applicable, as the amount of money cannot be infinity. Similar approaches can be found in famous monetary papers including Benhabib, Schmitt-Grohe, and Uribe (2001), Woodford (2003), and their proceedings.

In summary, some empirical and theoretical analyses urge for NIR, whereas some papers oppose NIR by emphasising the rationale behind ZLB. These historical observations and theoretical monetary debates motivate me to research NIR in contemporary Japan.

3.3 Why Consumption and Investment?

Consumption and investment are two key variables of interest to macro-economists. As explained in subsection 3.2, consumption is affected by household's reaction to interest rates, and investment is affected by firm's reaction to interest rates, which are being discussed in numerous studies.

For example, Lucas (1972), Lucas (1978), Lucas (1980), and some of his later works focused on consumption as the sole source of output; and Kiyotaki and Moore (1997) only discussed consumption and investment.

A quote from an economic theorist concludes my reason to focus on only two components:

> "If you model everything in the world, then there's nothing to show, because you can't solve it."
>
> — Polemarchakis (2018)

Indeed, given the fact that consumption and investment are the key variables of interest to economists, and difficulties of modelling increase once more components are considered in one paper, I choose to focus only on consumption and investment.

3.4 Why Building an Economic Model?

A further attempt to investigate consumption and investment via macroeconomic data is shown below.

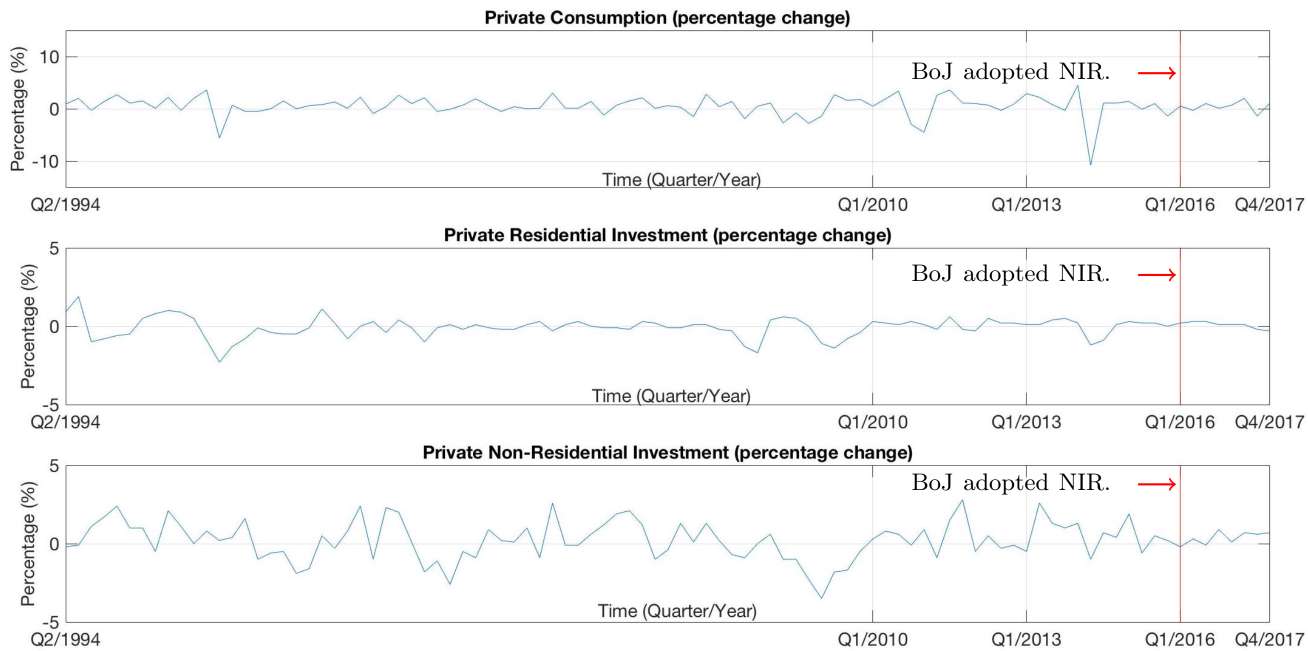

Figure 2: Plot of changes in private consumption and key components in investment (Seasonally Adjusted Real Quarter-to-Quarter percentage change) shows no significant variations or breaks after NIR being adopted.
Data Source: Cabinet Office (2018).

As plotted in Figure 2, no significant change in variables can be found post-2016. Moreover, during the past five years, Abenomics has resulted in fiscal stimulus and currency depreciation in Japan, which caused positive shocks to output (The Government of Japan 2018). So it may be hard to explicitly separate the effects by Abenomics post-2013 and the effects by NIR post-2016 within 8 observations.[9]

[9] Quarterly data from Q1/2016 to Q4/2017 means 8 data points for each variable.

The above illustrates one of the reasons not to take an econometric approach. More importantly, it is the lack of theoretical development on NIR in Japan that motivates me to write this paper by building an economic model. In subsection 3.1, I realised the lack of research in Japanese monetary issues, and the need for models that suit Japan, which I aim to provide. In subsection 3.2, I learnt about the current debates on NIR, which I wish to join.

Therefore, my approach is to build a Japan-specific economic model.

4 Data and Empirical Observations

4.1 Interest Rates

The observation in subsection 3.1 about Japanese commercial banks' inability to pass NIR to depositors motivates the following assumption.[10]

Preliminary Assumption 1. Write i_d be the deposit rate[11], i_c be the short-term interest rate. i_c is allowed to be negative, but i_d is not allowed to be negative.

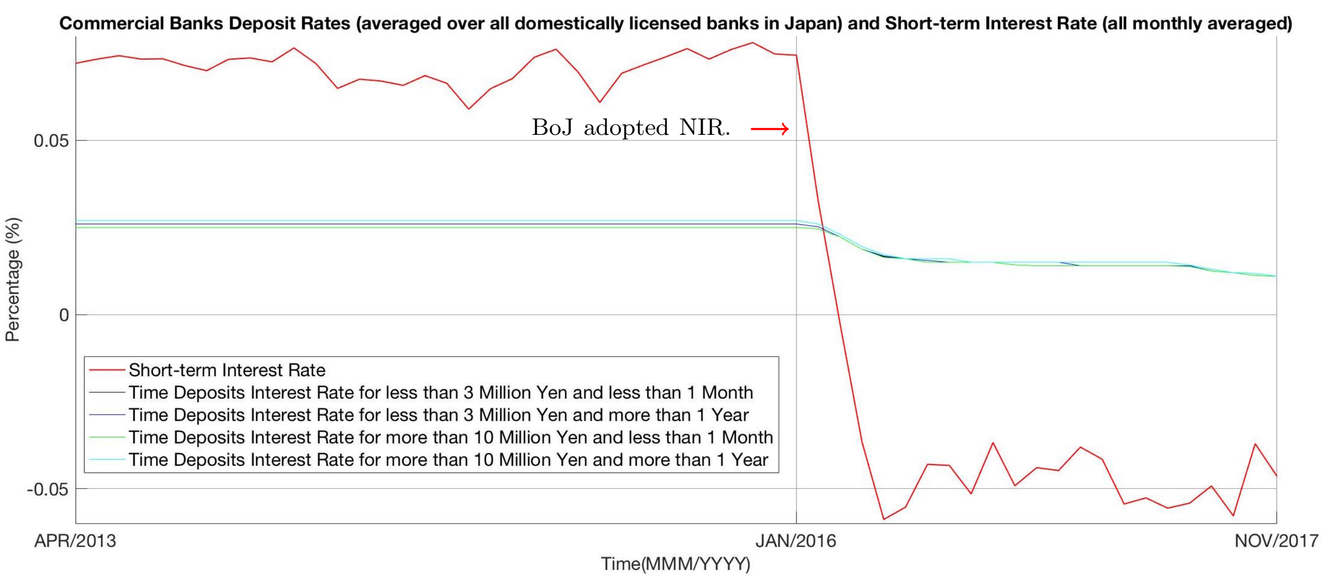

Figure 3: Various types of deposit rates fell, but still being positive after the implementation of NIR, despite short-term interest rate fell below zero.
Data Source: Bank of Japan (2018b).

Figure 3 shows a clearer dynamic of deposit rates before and after the rate cut. Such empirical observation motivates me to assume in my model that, deposit rate being higher than central bank rate before the rate cut, and lower than central bank rate after the rate cut.

[10] Note on i_c: according to Bank of Japan (2016), the "three-tier system" was adopted after NIR being implemented, meaning that commercial banks are offered three tiers of different interest rates for their accounts at BoJ. But for modelling simplicity, I assume there is only one negative interest rate being applied.
[11] i.e. the interest rate for deposit.

Certainly, the standard textbook-styled results hold in Japan as shown in Figure 3 and Figure 4, that borrowing rate is higher than deposit rate due to the risk premium (Howells and Bain 2008, pp.214-217).

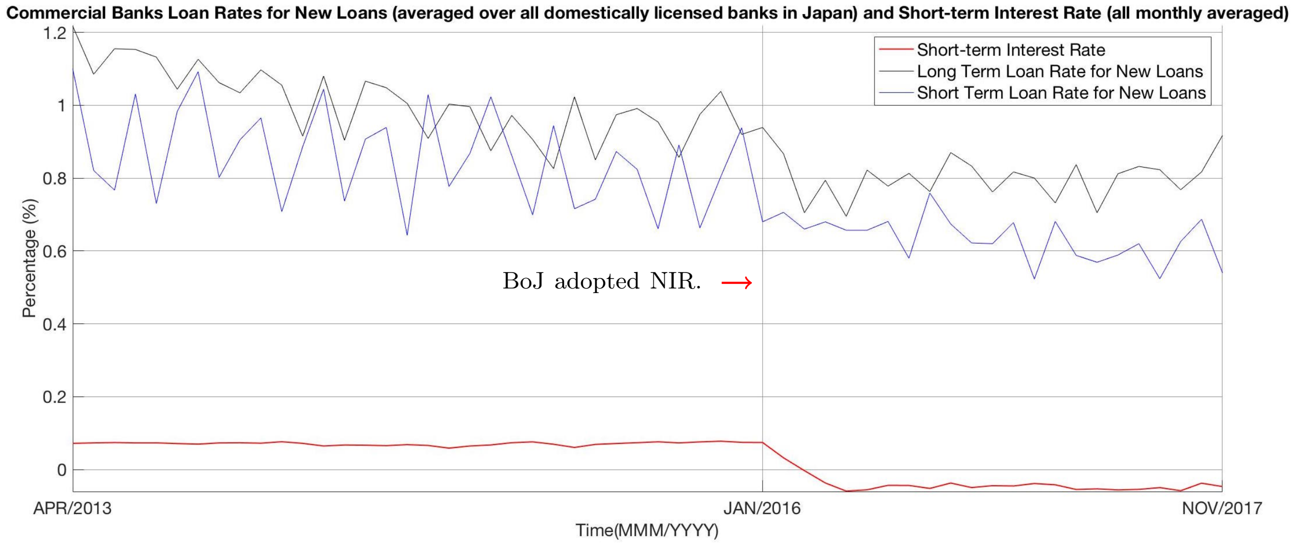

Figure 4: Borrowing rates slightly fell after the implementation of NIR.
Data Source: Bank of Japan (2018b).
Note on definition: I define borrowing rate as the interest rate for borrowing (or lending from commercial banks' viewpoint), and loan rate to be the same as borrowing rate.

Based on the above summaries from Figure 3 and Figure 4, I extend Preliminary Assumption 1 to the followings:

Preliminary Assumption 2. Write i_b to be the borrowing rate.
$\forall X \in \{i_d, i_c, i_b\}$, write $X(0)$ to be the value of X before NIR being implemented, and $X(1)$ to be the value of X after NIR being implemented. Then

$$i_b(0) > i_c(0) > i_d(0) \geq 0 \tag{2a}$$
$$i_b(1) > i_d(1) \geq 0 > i_c(1) > -1 \tag{2b}$$
$$i_b(0) \geq i_b(1) \tag{2c}$$
$$i_d(0) \geq i_d(1) \tag{2d}$$

Equation 2a and Equation 2b summarise the ordering of borrowing, deposit, and short-term interest rates before and after the rate cut. Notice the last part of Equation 2b — $i_c(1) > -1$, meaning central bank interest rate cannot fall below -100%. [12] Equation 2c covers the small drop in borrowing rates after the rate cut, and Equation 2d covers the same dynamics for deposit rates.

[12] Of course, the worst scenario is the entire deposit disappeared — that is -100%, but anything lower than that, e.g. -101% has barely any meanings.

Another effect of NIR is on bond yields.

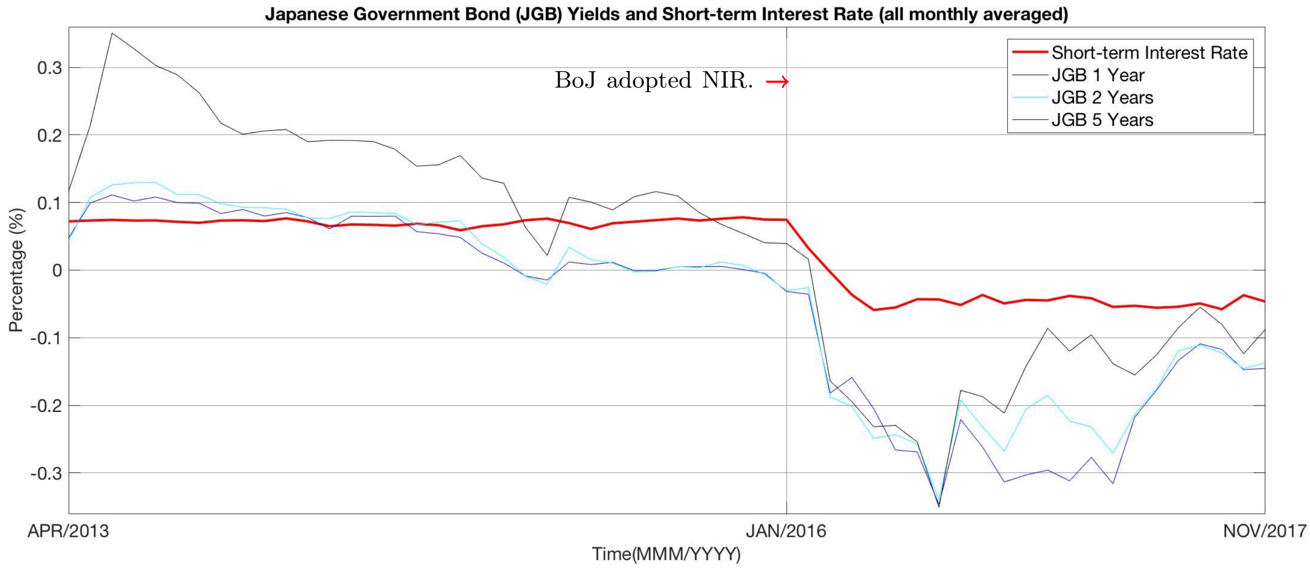

Figure 5: JGB yields fell below zero after the implementation of NIR.
Data Source: Bank of Japan (2018b) and Bloomberg Terminal (2018).

Figure 5 shows, some JGB holders are paying to hold bonds after NIR was adopted. Indeed, commercial banks are paying to hold bonds — as of September 2017, Japanese commercial banks hold a summation of 199.1 trillion yen of JGBs, which is 20.3% of the total JGB amount (Ministry of Finance, Japan 2018).[13]

The above data imply, post-2016, Japanese commercial banks are unable to avoid holding some amount of bonds which yield negative rates. I cover this in my model by assuming commercial banks facing a "minimum floor" on their allocation of assets:[14]

Preliminary Assumption 3.
There exists a minimum proportion of commercial banks' assets where interest rate being applied at i_c.

4.2 Commercial Banks, Loans, and Deposits

Compared to the scenario where JGB yield is positive, ceteris paribus, holding negatively yielded JGBs decreases profit. This motivates me to ask: are commercial banks' profits being damaged by NIR?

Profit data from different banks give different answers to my question[15], whereas Nakano (2016, p.178) gave a logical comment — bank profits are decreased by the further diminishing lending yields due to NIR. Furthermore, Barwell (2016, p.43) suggested, NIR may squeeze bank profitability and may tighten the banks' willingness to lend. So, what does the data show?

[13] Despite not all of the 199.1 trillion yen of JGBs are negatively yielded, a significant proportion of them are. Further details, including the T-Bill holders breakdown, can be found at Ministry of Finance, Japan (2018).

[14] Note: I implicitly assume the i_c to be the same as JGB yield in my model for simplicity. This is based on the theoretical reasons that bond yield is closely related to central bank rate (Howells and Bain 2008, pp.214-222), and Figure 5 shows so.

[15] See Lee (2018) and Nikkei Asian Review (2018) for media sources.

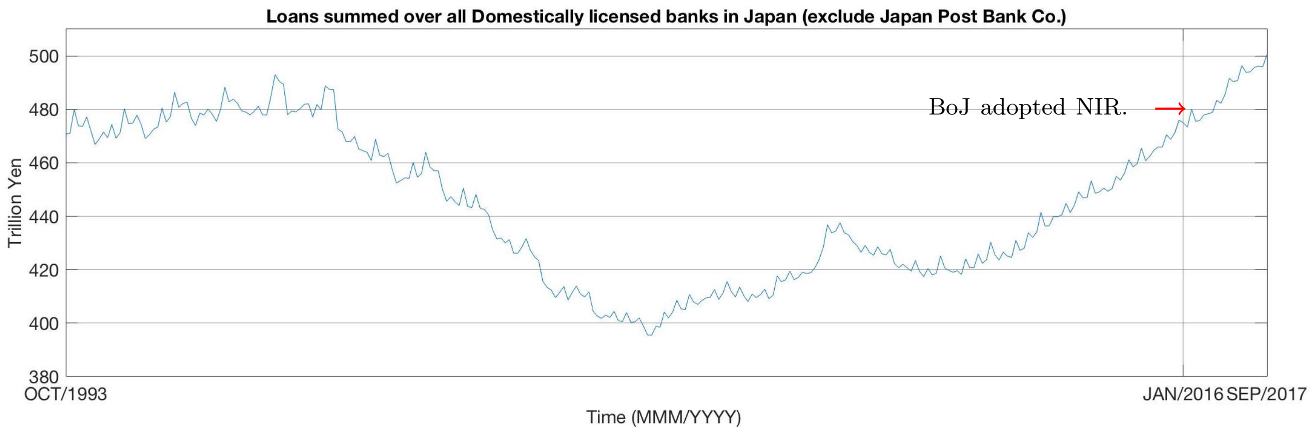

Figure 6: Total loans in Japan generally increased post-2010. No major break or reversion after NIR being adopted.
Data Source: Bank of Japan (2018b).

One may comfortably conclude from Figure 6 that banks' lendings increase, and no significant differences were induced by NIR. However, the deposit data may change one's mind.

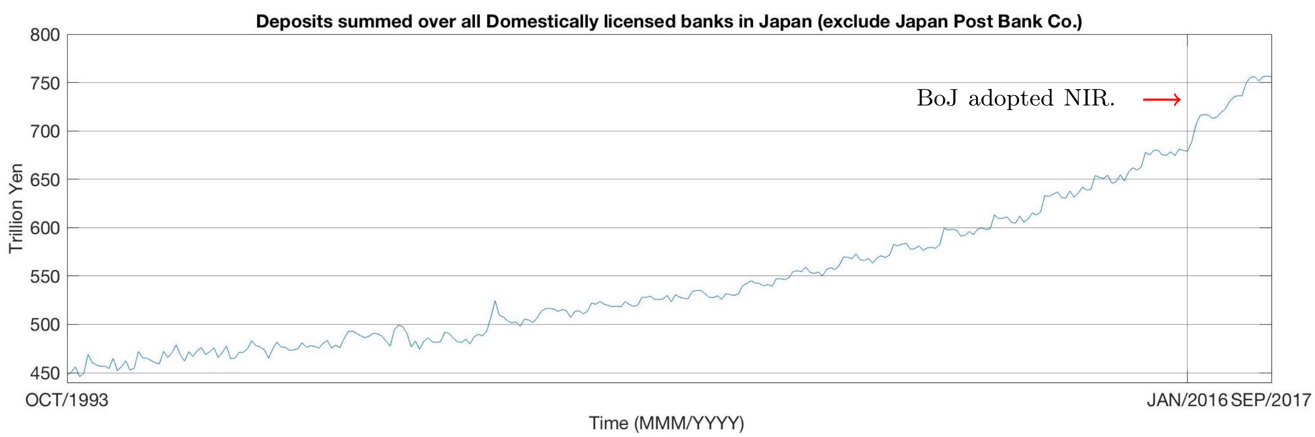

Figure 7: Total deposits in Japan have been steadily increasing for more than two decades. No major reversion after NIR being adopted.
Data Source: Bank of Japan (2018b).

So, the increase in loans may just be constantly pushed by the increase in deposits as shown in Figure 7, due to the long-lasting expansionary monetary policies since the 1990s.[16] This motivates me to calculate the loans-over-deposits ratio (LOD) by defining

$$LOD := \frac{L}{D}$$

where L is the total loans and D is the total deposits.

[16] One may recall the historical review in subsection 3.2.

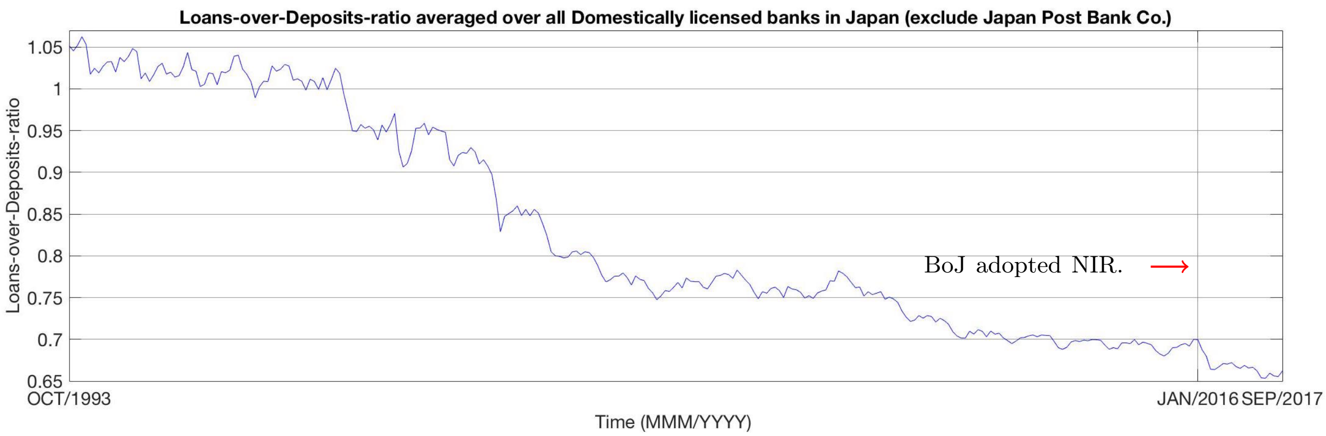

Figure 8: *LOD* decreases throughout the past two decades, and NIR has no positive impact on it. Data Source: Bank of Japan (2018b).

Figure 8 suggests, on average, the proportion of deposits being lent out has been decreasing and NIR does not help to push it back. This motivates me to make the following conjecture:

Conjecture 1. *NIR leads commercial banks to decrease lendings.*

Having reviewed the stories in contemporary Japan, the next section reviews some theoretical papers that help to build and solve my model.

5 Literature Review on Technical Papers

5.1 Microfoundations

> " The Chicago way of developing macroeconomics was by developing microeconomics so that a certain macroeconomic story could be told. "
>
> — Lucas (2013, p.x)

The above quote shows my theme for this subsection. In my opinion, having a microeconomic foundation enables me to interpret individual's behaviour and to explain the macroeconomic numbers. This would both defend my result, and give me the chance to add Japanese elements in the model. Moreover, if I follow standard macroeconomics textbooks, e.g. Carlin and Soskice (2015), it would be hard to complete either of the above two tasks by manipulating the three equations in "IS-PC-MR" system.

Debates on microfoundations carry on as described by Cencini (2005) and Duarte and Lima (2012), but I stand on building a microfoundation, in particular, to develop a microeconomic model on credit mechanism so that the macroeconomic story on consumption and investment can be told.

5.2 Economic Modelling

In general, Ljungqvist and Sargent (2012) provided fundamental settings for an agents-based economy, with extensions to Dynamic Stochastic General Equilibrium (DSGE) models. Some microeconomic concepts from Knight (1921) and Mas-Colell, Whinston, and Green (1995) are also useful during the construction of my model. Most of the main constructions and proofs in my model rely on utility theory, thus theoretical studies on utility theory[17] are useful.

Specifically, the individuals' utility functions contain labour supply in Lucas (1972, p.106). Such types of settings link individual behaviours to inflation and other key macroeconomic components, which New Keynesian Monetary Models and other DSGE Models draw.[18] However, it also brings complications into the model, with complex mathematical problems where multiple times of approximations[19] are necessary to obtain numerical solutions.

I decide not to include labour supply in utility functions after the above observation. This is because the labour market, despite being important from a generic macroeconomic perspective, is not the key question in my paper; and I plan not to use any approximation to obtain the closed-form solutions.

Another simplification I make is in regards to cash. From Walsh (2010, pp.34-129), cash holding shall be included in utility or budget constraints of individuals, then interesting analysis on liquidity can be made. However, approximations are again necessary to solve the model, due to the complexity. Thus I decide to keep the utility functions as simple as possible by putting consumption of goods and discount factors as the only sets of variables in individuals' utility functions. Similarly, cash holding is not involved in budget constraints in my model.

Noticeably, Woodford (2003, pp.61-72) also made the above simplifications in the early part of his models.

The remaining part of this subsection reviews a famous paper from Schmitt-Grohe and Uribe (2009), which partly extends the model from Woodford (2003) to explain NIR. In Schmitt-Grohe and Uribe (2009, pp.89-90, 92-93, 95-97), the central bank was enabled to set any NIR, and the household budget constraint was the same function for all interest rates set by the central bank. These imply, the policy-induced NIR is assumed to be exactly, or at least a close proxy for, the interest rates households face in their budget constraint. But this conflicts with the empirical facts in Japan.

In Japan, after NIR was imposed, households still faced non-negative interest rates[20], thus the policy-induced interest rate is no longer a close proxy for the interest rates that households face in their budget constraint — more accurately, the policy-induced rate fell below zero whereas deposit rates remained at ZLB.[21]

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[17] e.g. Barbera, Hammond, and Seidl (1998), Barbera, Hammond, and Seidl (2004), and Greco, Matarazzo, and Slowinski (2004).

[18] See Walsh (2010, pp.329-378) and Gali (2015) for further details.

[19] e.g. "linearisation" in Walsh (2010, pp.336-352).

[20] By the observations in subsection 3.1.

[21] See Figure 3 for further details.

The above argument illustrates an empirical shortcoming of Schmitt-Grohe and Uribe (2009), when explaining NIR in Japan. Such a shortcoming also warns me not to conduct the macroeconomic analysis directly without any support from the microeconomic mechanism on NIR. Additionally, the fact that Japanese commercial banks did not set negative interest rates to households motivates me to get a closer look at commercial banks and credits, to address the macroeconomic question of my paper.

5.3 Commercial Banks and Credits

Textbooks such as Howells and Bain (2008) and Matthews and Thompson (2014) provide the fundamentals to model commercial banks.

Motivated by Figure 8, I found Poole (1968) who analysed the Loans-Deposits relationship. Poole (1968) used a short-term model explaining the optimal cash holding for a commercial bank, which depends on multiple variables including the interest rate offered by the central bank[22], discount rate, and subjective probability distribution on net deposit accretion. However, if interest rates were allowed to be negative, then the model needs to be modified. I include some of the variables that affect commercial bank's balance sheet position in Poole (1968) into my model.

Graziani (2003, pp.114-128) showed the role of the financial market in an economy by including different types of interest rates in his model. I found this suits my need as shown in Preliminary Assumption 2 — members of the economy face different types of interest rates, which shall be covered by my model.

Goodfriend and McCallum (2007) continued the work from Poole (1968), and drew an outline for the commercial bank's balance sheet, then found solutions in the macroeconomic model set by Woodford (2003, pp.299-311). However, ZLB was assumed, thus the result did not address NIR. I take the initiative of Goodfriend and McCallum (2007) and consider a view from the balance sheet.

Adrian and Shin (2010, pp.8-12) modelled a balance sheet of a leveraged investor, then solved the optimal holding of securities. I make some adjustments towards Adrian and Shin (2010) to reframe the balance sheet from a leveraged investor to a more "defensive"[23] Japanese commercial bank.

Stiglitz and Weiss (1981, pp.393-398, 401-402) made an important microeconomic conjecture on credit rationing — interest rate affects the nature of credit market transaction, thus credit market may not clear. There is, however, a lack of justification from an assumption by Stiglitz and Weiss (1981, p.397) — the non-linear relationship between the supply of loan and bank's profits.

Adrian and Shin (2013) used Value-at-Risk (VaR) model to explain the non-linear relationship between bank's asset and lending[24]. But the VaR model was derived based on the balance sheet of a leveraged

[22] Federal Funds Rate in the literature.

[23] As Nakano (2016, p.177) described. One could also refer to Figure 8 — up-to-date data suggests that, in Japanese commercial banks, loans are less than 70% of deposits on average, meaning no debt is required for the banks, for some extra loan issuance, i.e. "no leverage" in the framework of Adrian and Shin (2010).

[24] Traded securities in the literature.

investor from Adrian and Shin (2010), so it would be inconsistent to adopt the VaR model to a Japanese commercial bank, where balance sheet structure largely differs from a leveraged investor, as explained previously. Therefore, I take the idea, but not the whole model from Adrian and Shin (2013) and use utility theory to analyse the non-linear relationship between bank's asset and lending.

Compared to Stiglitz and Weiss (1981), Basu (1991, pp.15-34) made a simpler microeconomic model. The bank profit maximisation with credit rationing results in Basu (1991) are partially used in my model.

Willman (1981) and Kahkonen (1982) added commercial banks' objectives as part of the macroeconomic systems, and showed the macroeconomic impact of credit rationing due to some credit market restrictions. My paper makes a similar structure to their process on model building and solving. One of the key assumptions from Willman (1981, p.11) was that loan and deposit rates being set institutionally and independently of the amount of credit. I take this assumption which simplifies my model.

Kiyotaki and Moore (1997) also provided macroeconomic implications with an economic model centred on credit flow. However, "free-trade" was assumed in the credit market — it was frictionless and there was no place for commercial or central banks to step in. This discourages me from using their credit market settings, but I take their agents-centred goods market settings, e.g. agents consume the fruit, which is a single non-durable commodity.

Lastly, Masi et al. (2011) studied empirically on Japanese credit network, and concluded that many firms have credit networks with only 1 or 2 commercial banks. This motivates me to assume, there is only one commercial bank in my model, which avoids further microeconomic complications on market structures.

5.4 Mathematics

This subsection briefly reviews useful mathematics textbooks that help me to build and solve my model.

Basic knowledge of probability is acquired from Pitman (1993) and Hogg, McKean, and Craig (2012), which are sufficient in model building.

Some maths-for-economists textbooks including Bagliano and Bertola (2007), Sydsaeter et al. (2008), and Stachurski (2009) offer basic knowledge on optimisation-related topics with practical approaches, but their theoretical results are insufficient for me.

Clapham and Nicholson (2009) gave fundamental definitions to mathematical objects used in my model building and solving. Nikaido (1970) provided easy applications to optimisation problems. Vohra (2005) offered further knowledge on convexity and non-linear programming, and purer mathematics textbooks[25] give a richer theoretical framework for optimisation. Additionally, Sutherland (2009) gave a deeper introduction to topologies, followed by Milnor (1997), which presented further useful results in higher dimensional differentiation with topological settings.

[25] e.g. Beck (2014).

Some examples in my solution require solving polynomial equations. Garling (1986) offered excellent works on Galois Theory, which provides theoretical arguments for solving roots of a polynomial.

Having reviewed the empirical observations and technical literature, I now present my model.

Part II

The Model

Overview: section 6 sets a model that covers agent's Utility Maximisation Problems (UMP). Then, section 7 uses theoretical mathematics to solve UMP, followed by numerical examples in section 8.

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6 Economic Settings

6.1 Introduction by One Story

This subsection introduces the intuition behind my model without any use of mathematics.

Consider an agent-based economy with a Land Registry, a Central Bank, and a Commercial Bank ($ComB$).

Farmers produce and consume a single non-storable good — banana. Investment can be made by buying land from the Land Registry so that production increases next period. Production is risky, thus $ComB$ needs to consider credit default risk while lending. Central Bank decides on the nominal interest rate (i_c) on Bonds, which is assumed to be a risk-free asset.

I set the model which includes Japanese institutional structure of the banking system as reviewed in Part I, including $ComB$ being unable to set negative interest rates to households and unable to avoid holding some amount of Bonds. Then I show the following.

Suppose the Central Bank cuts i_c from positive to negative, then $ComB$ loses some profits. Thus $ComB$ has less confidence in taking risks, and may not lend as much as it did before the rate cut. Finally, some agents face a decrease in consumption and investment because they have tighter budgets due to their borrowings not being granted.

6.2 Farmers

Setting 1. General.

(a) Credit Network: *ComB* has credit network with all of the farmers, and there is no credit network amongst farmers.

(b) Period: there are two periods — Period 1 and 2.

Setting 1(a) implies that, for farmers, the only possible way to make deposits and borrowings is through the commercial bank. Setting 1(b) simplifies my model by using finite periods.

Setting 2. Price and Inflation.

(a) The price of banana is $P_1 > 0$ per unit at period 1, and $P_2 > 0$ per unit at period 2. Inflation (π) is defined as below.

$$\pi := \frac{P_2 - P_1}{P_1}$$

(b) Let every farmer be the taker of P_1 and P_2.

Setting 2(a) introduces some usual notations, and Setting 2(b) restricts farmer's choice variables, which are supporting the UMP settings later.

Setting 3. Production of Goods and Investment.

(a) The farming of banana only requires land as the sole capital input, and the production of banana is independent of labour input, so there is no labour market.

(b) The economy has a sufficiently large amount of unharvested land owned by the Land Registry, and investment is required from farmers to buy those land at a fixed price for future production. Assume the land is non-tradable amongst farmers, and farmers cannot sell them back to the Land Registry.

One may feel discomfort on Setting 3(a), but this is set to prevent the complication of the model.[26] Similarly, Setting 3(b) may be blamed by free-market economists, but is set to avoid modelling difficulties.

[26] One could also refer back to subsection 5.2 for explanations.

Setting 4. Farmer's Utility Function. [27]

Write $C_{t;j} \in \mathbb{R}_{\geq 0}$ to be the consumption (measured by units of bananas) at period t for farmer j. Suppose the utility of farmer j at a given period t can be modelled by a utility function

$$u_j : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad C_{t;j} \mapsto u_j(C_{t;j})$$

such that

$$u_j \in C^2(\mathbb{R}_{\geq 0}, \mathbb{R}) \tag{3a}$$

$$\forall x \in \mathbb{R}_{\geq 0}, \quad u_j'(x) > 0 \quad \text{and} \quad u_j''(x) < 0 \tag{3b}$$

The above introduces a standard risk-aversive (strictly concave) definition of utility functions. Now, I model the risk of production by setting a weather-dependent production for period 2. Notice that, period 1 is with no risk at all.[28]

Setting 5. Weather and Common Knowledge.

(a) Let weather at period 2 be a binary outcome for each farmer's land. Write X_j to be the weather on the land of the farmer j at period 2. One outcome is good, write as $X_j = G_j$, with probability $p_j \in (0, 1)$; and the other outcome is bad, write as $X_j = NG_j$, with the remaining probability $1 - p_j$. Assume the weather on each land is pairwise independent.

(b) Assume there is common knowledge between farmers and $ComB$ at the start of period 1, and let this knowledge be broad enough such that there are weather-induced risks, but no uncertainty.[29]

To intuitively include the weather-induced risk as part of the common knowledge, one can suppose at the start of period 1, the Land Registry solely and convincingly produces a probabilistic weather forecast for period 2, and suppose the Land Registry is telling the truth.

By the above settings, I conclude the production of Banana in Period 2 to be dependent on investment in Period 1 and weather in Period 2, thus write the mathematical model overleaf.

[27] See section 12 for the definition of twice continuously differentiable functional set (C^2).

[28] So, one could treat period 1 production as "endowment".

[29] See Mas-Colell, Whinston, and Green (1995, p.207) for the distinction between risks and uncertainty. In my opinion, "Uncertainty" refers to the concept of "unknown unknown" — the collection of scenarios where one cannot attach the objective probability to; whereas "risks" refers to "known unknown" — the collection of scenarios where one knows accurately the objective probability of. Also, see Knight (1921) for further discussions.

Setting 6. Consumption and Production Functions in Period 2.
For each farmer j, write $I_j \in \mathbb{R}_{\geq 0}$ as the investment at period 1, define the outcome set as

$$\Omega_j = \{G_j, NG_j\} \tag{4a}$$

consumption function as

$$C_{2;j} : \Omega_j \oplus \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \qquad (X_j, I_j) \mapsto C_{2;j}(X_j, I_j) \tag{4b}$$

and production function as

$$F_{2;j} : \Omega_j \oplus \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \qquad (X_j, I_j) \mapsto F_{2;j}(X_j, I_j) \tag{4c}$$

such that

$$F_{2,j}(G_j, y) \geq F_{2,j}(NG_j, y) \quad \forall y \in \mathbb{R}_{\geq 0} \tag{4d}$$

$$\forall X_j \in \Omega_j, \quad \text{if} \quad f(y) := F_{2,j}(X_j, y), \quad \forall y \in \mathbb{R}_{\geq 0} \quad \text{then} \quad f \in C^2(\mathbb{R}_{>0}, \mathbb{R}) \cap C^0(\mathbb{R}_{\geq 0}, \mathbb{R}) \tag{4e}$$

$$\forall X_j \in \Omega_j, \forall y \in \mathbb{R}_{\geq 0}, \quad \frac{\partial F_{2,j}(X_j, y)}{\partial y} > 0 \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial F_{2,j}(X_j, y)}{\partial y}\right) < 0 \tag{4f}$$

Note (a): it might be more intuitive to model production as a function of land, but since lands are bought at a fixed price, it is equivalent to say production as a function of investment.
Note (b): I_j is measured by monetary amount, whereas $C_{2;j}$ and $F_{2;j}$ being measured by units of banana.

Equation 4a, Equation 4b, and Equation 4c declare the variability of consumption and production in period 2, against weather and investment, as previously introduced in Setting 5. Equation 4d makes a clearer distinction between good and bad weather — production under good weather yields more[30] banana than production under bad weather. For each given weather, Equation 4e and Equation 4f rule the decreasing-marginal-return (strictly concave) property into the production function.

[30] Here, "more" is equivalent to $\geq$ in maths.

Setting 7. Farmer's UMP.

Every farmer j faces UMP at period 1 as

$$\max_{C_{1;j}\geq 0 \text{ and } C_{2;j}\geq 0} \left\{ \mathbb{E}\Big[u_j(C_{1;j}) + \beta_j u_j(C_{2;j}) \Big] \right\} \tag{5a}$$

subject to

$$C_{1;j}P_1 = F_{1;j}P_1 + B_j - I_j \tag{5b}$$

$$C_{2;j}P_2 = \begin{cases} 0 \quad \text{ if } F_{2;j}(NG_j, I_j)P_2 \leq B_j(i_{\#}+1) \text{ and } X_j = NG_j \\ F_{2;j}(X_j, I_j)P_2 - B_j(i_{\#}+1) \quad \text{otherwise} \end{cases} \tag{5c}$$

where

$$i_{\#} = \begin{cases} i_b, & \text{if } B_j > 0 \\ i_d, & \text{if } B_j \leq 0 \end{cases} \tag{5d}$$

Notation	Meanings	Domains	Remarks
β_j	Constant discount factor of farmer j	$\beta_j \in (0, 1]$	
$F_{1;j}$	Output of farmer j at period 1 from farming	$F_{1;j} \in \mathbb{R}_{>0}$	
B_j	Borrowing or Deposits of farmer j at period 1	$B_j \in \mathbb{R}$	$B_j > 0$ if borrow, $B_j \leq 0$ if save
i_b	Borrowing rate at period 1	$i_b \in \mathbb{R}_{\geq 0}$	
i_d	Deposit rate at period 1	$i_d \in \mathbb{R}_{\geq 0}$	

Table 1: Variables

Note: in some literature, "$\leq$" is used instead of "$=$" in Equation 5b and Equation 5c. But these two are equivalent because of the strictly increasing property of the utility function — $\forall x \in \mathbb{R}_{\geq 0},\ u'_j(x) > 0$ as per Setting 4.

Equation 5a is a standard expected utility maximisation objective. Equation 5b means, in monetary value, the total production plus borrowings minus investment shall be equal to consumption. Equation 5c means, if bad weather happens and the payables exceed the production, then consume nothing[31]; otherwise, consumption is the production minus payables[32]. Equation 5d highlights the fact that farmers face borrowing rate when borrowing, and deposit rate when saving.

[31] Because all of the productions would be collected by the bank, which act as a partial payback to the debt. One may call this scenario as "Credit Default".

[32] Or plus receivables in case saved in the first period.

Equation 5c also underlines another intuitive information. The lower half of Equation 5c with $C_{2;j} \geq 0$ implies $F_{2;j}(G_j, I_j)P_2 \geq B_j(i_{\#} + 1)$. This is reasonable, because otherwise, payables in period 2 exceed the production in all weather outcomes[33], meaning the farmer has no chance to pay back the full amount — thus the $ComB$ has no chance to receive the full payback under such huge borrowings, so $ComB$ would not lend out such amount at start, under the common knowledge settings.

Finally, I link the farmer's consumption and investment behaviour to the commercial banks by the following setting on Credit Market.

Setting 8. Setter and Taker in Credit Market.

(a) Let every farmer be the taker of i_b and i_d .

(b) Let every farmer be the setter on the amount of money to save.

(c) If a farmer applies an amount of money to borrow, then the farmer is a taker of the decision from $ComB$, which is either credit granted or not.

Setting 8(a) rules out interest rates as choice variables for farmers. Setting 8(b) and (c) underline a credit mechanism: the farmer applies the credit to $ComB$, and if $ComB$ grants the credit, then the farmer gets the credit. Otherwise, the farmer faces an extra credit restriction $B \leq 0$ due to its application not being granted, i.e. the farmer faces a rationed credit.

Figure 9 describes the above mechanism more precisely.

[33] Indeed, once $F_{2;j}(G_j, I_j)P_2 < B_j(i_{\#} + 1)$, then $F_{2;j}(NG_j, I_j)P_2 < B_j(i_{\#} + 1)$ because of Equation 4d.

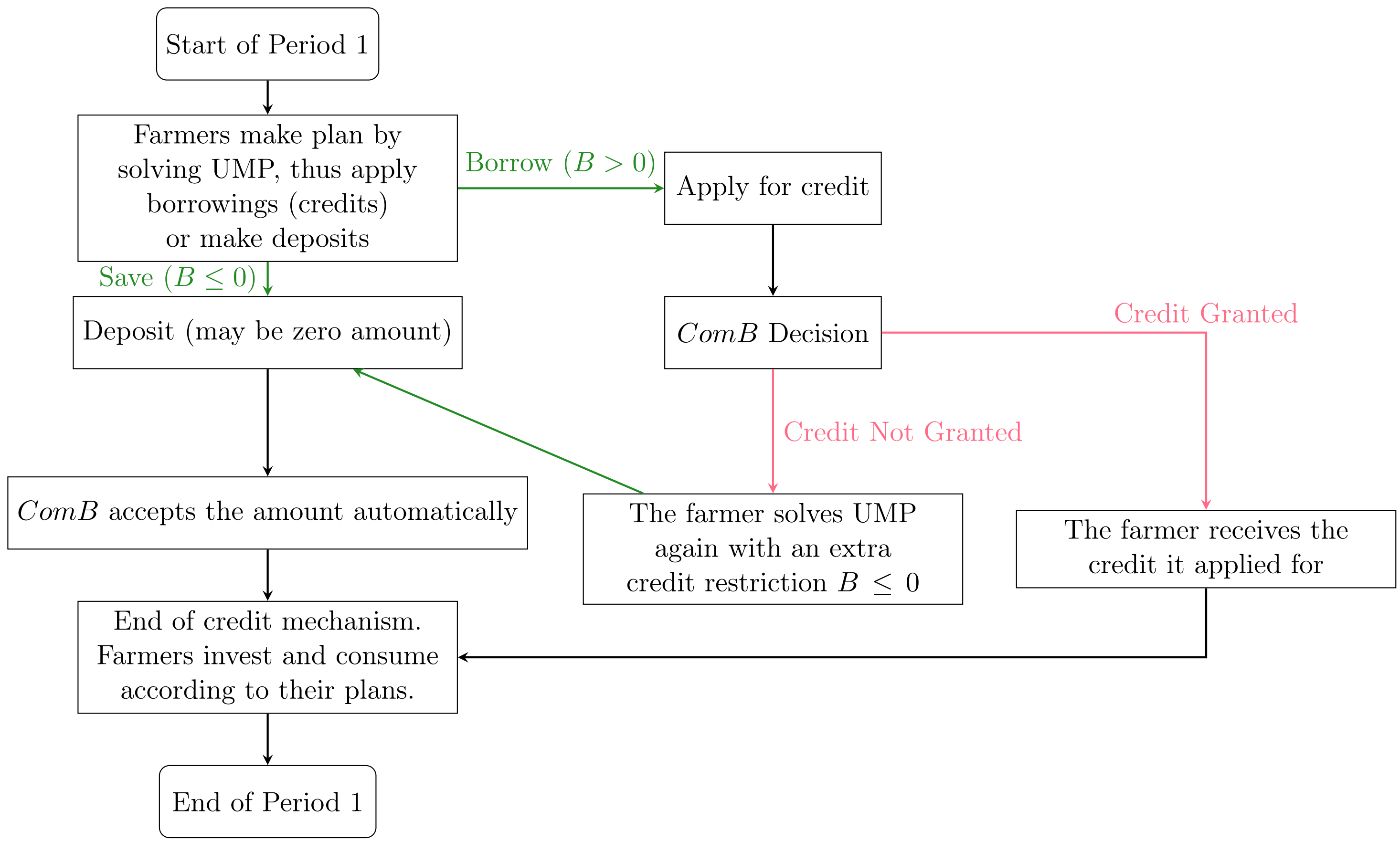

Figure 9: Flowchart for credit mechanism in Period 1

In Figure 9, all black arrows are either set in this subsection or automatically assumed. All green coloured arrows are modelled in this subsection. So, next subsection models the "$ComB$ Decision", i.e. the Cherry Blossom coloured arrows.

6.3 Commercial Bank

Setting 9. Commercial Bank.

(a) The utility of $ComB$ is measured by its utility on profit from Period 1, and can be modelled by a utility function u_{ComB}.

(b) $ComB$ is a taker of i_b , i_c , i_d, with further settings in Preliminary Assumption 2.

(c) If $B_j \leq 0$, then $ComB$ takes the resulting deposit; otherwise if $B_j > 0$, then $ComB$ either agrees to lend the exact amount B_j, or not to lend anything.

(d) $ComB$ has its balance sheet as shown in Figure 10, Cash (A_0) and Government Bonds (A_1) are assumed to be risk-free, and Equity (L_1) does not change during Period 1.

(e) $ComB$ maximises its expected utility subject to some legal constraints and market conditions. The legal constraints include that $ComB$ would always be able to return all of the depositors' money. The market conditions include Preliminary Assumption 3 and the full knowledge of Figure 9 with Farmer's UMP.

Balance Sheet of the *ComB*

Assets	Liabilities
Cash (A_0)	Deposits from Farmers (L_0)
Bonds (A_1)	
Lendings to Farmers (A_2)	Equity (L_1)

Figure 10: $ComB$ Balance Sheet

Setting 9(a), (b), and (e) not only outline the $ComB$'s UMP, but also link the preliminary assumptions in Part I to my model. Setting 9(c) makes the $ComB$ setting coherent with Figure 9, and Setting 9(d) makes the model simpler.

Setting 10. Accounting Principle of Balance Sheet.

Write n to be the number of farmers in the economy, and for each farmer j, write B_j^* to be the amount of lending that $ComB$ agrees to deposit or lend.

$\forall X \in \{L_0, L_1, A_0, A_1, A_2\}$, write, at period 1, the value of X if i_c is positive as $X(0)$, otherwise if i_c is negative as $X(1)$. Write the value of X at period 2 as $X(2)$.

$$\forall t \in \{0,1,2\}, \quad L_0(t) = \sum_{j=1}^{n} -B_j^* \mathbb{1}[B_j^* \leq 0] \tag{6a}$$

$$\forall t \in \{0,1\}, \quad A_0(2) = A_0(t) + \sum_{j=1}^{n} (i_d) B_j^* \mathbb{1}[B_j^* \leq 0] \tag{6b}$$

$$\forall t \in \{0,1\}, \quad A_1(2) = (1+i_c) A_1(t) \tag{6c}$$

$$\forall t \in \{0,1\}, \quad A_2(t) = \sum_{j=1}^{n} B_j^* \mathbb{1}[B_j^* > 0] \tag{6d}$$

$$A_2(2) = \sum_{j=1}^{n} \left(\Big((1+i_b) B_j^* (1 - \mathbb{1}[Default_j]) + F_{2;j}(NG_j, I_j) P_2 \mathbb{1}[Default_j] \Big) \mathbb{1}[B_j^* > 0] \right) \tag{6e}$$

where the default indicator is defined as:

$$\mathbb{1}[Default_j] = \begin{cases} 1, & \text{if } (1+i_b) B_j^* > F_{2;j}(NG_j, I_j) P_2 \text{ and } X_j = NG_j \\ 0, & \text{otherwise} \end{cases} \tag{6f}$$

Equation 6a links deposits to Farmer's savings. Equation 6b means, cash in period 2 is cash in period 1 deduct interest payables to depositors. Equation 6c means, bonds in period 2 is the bonds in period 1 plus interest receivables from [34] Central Bank.[35] Equation 6d links the period 1 lendings to the Farmer's Borrowings. Equation 6e and Equation 6f link the period 2 receivables from lendings to Farmer's situation in period 2 — if a farmer defaults, then all the production from that farmer becomes a partial repayment to debt, otherwise full repayment of debt plus interest. This corroborates with Setting 7.

[34] Or "payables to" if i_c is negative.

[35] One may say bonds are issued by the government and interests being paid by government. But I do not set government in my model, so Central Bank is the only appropriate identity to put here.

Setting 11. Utility Function of $ComB$.
Given $t \in \{0, 1\}$ and $L_1(t)$, define

$$u_{ComB} : [-L_1(t), +\infty) \to \mathbb{R} \qquad L_1(2) - L_1(t) \mapsto u_{ComB}(L_1(2) - L_1(t)) \tag{7a}$$

such that

$$u_{ComB} \in C^2\bigg([-L_1(t), +\infty), \mathbb{R}\bigg) \tag{7b}$$

$$\forall x \in [-L_1(t), +\infty), \quad u'(x) > 0 \quad \text{and} \quad u''(x) < 0 \tag{7c}$$

In Equation 7a, an underlining assumption[36] is $L_1(2) \geq 0$. This coheres with Setting 9(e), because otherwise "negative equity" would mean $ComB$ being unable to payback parts of the depositor's money, which conflicts with Setting 9(e).

Similar to Setting 4, the strict concavity of utility function is imposed by Equation 7c, making $ComB$ to be a risk-aversive identity, rather than a gambler.

Setting 12. UMP of $ComB$.
$ComB$ faces the following maximisation problem.
For a given $t \in \{0, 1\}$ and given constants $L_1(t) > 0,\ \lambda_0,\ \lambda_1 \in (0, \frac{1}{2})$,

$$\max_{\{B_j^*\}_{j=1}^n, A_0(t), A_1(t)} \mathbb{E}[u_{ComB}(L_1(2) - L_1(t))] \tag{8}$$

subject to

$$A_0(t) \geq \lambda_0(L_0(t) + L_1(t)) \tag{9a}$$

$$A_1(t) \geq \lambda_1(L_0(t) + L_1(t)) \tag{9b}$$

Finally, Equation 8 represents mathematically the Setting 9(e). Equation 9a and Equation 9b rule in the minimum cash and bond holdings for $ComB$, which also realise Preliminary Assumption 3.

[36] This is induced by $L_1(2) - L_1(t) \in [-L_1(t), +\infty)$.

7 Methodologies: General Solution and Functional Specifications

Note on notations and proofs: most of the standard notations are taken from textbooks mentioned in subsection 5.4, with some clarifications made in subsection 12.1. Proofs are located in section 13.

7.1 Farmer's UMP

Setting 13. Bounded inflation rate and Real Interest Rates.

$$|\pi| < 1 \tag{10a}$$

$$r_x := \frac{1+i_x}{1+\pi} - 1 \quad \forall x \in \{b, d\} \tag{10b}$$

$$r_\# := \frac{1+i_\#}{1+\pi} - 1 \tag{10c}$$

As I am not interested in hyperinflation or hyperdeflation, I assume Equation 10a, that inflation rate being always less than 100% and greater than −100%. Equation 10b and Equation 10c provide notational simplicity later in the solutions. [37]

7.1.1 General Solution

Setting 14. Farmer's Expectation.
Given a farmer j,

$$\mathbb{E}\Big[u_j(C_{1;j}) + \beta_j u_j(C_{2;j})\Big] = u_j(C_{1;j}) + \beta_j\Big(p_j u_j(C_{2;j}(G_j)) + (1-p_j)u_j(C_{2;j}(NG_j))\Big) \tag{11}$$

The above is a standard realisation for binary random-variables, which enables UMP being solved explicitly.

[37] r can be called as real interest rate, see the original definition by Fisher (1896).

Lemma 1. *Expected Utility Function Rewrite.*
Define

$$U_j : W_j \to \mathbb{R}$$

where

$$W_j := \left\{ (B_j, I_j) \in \mathbb{R} \oplus \mathbb{R}_{\geq 0} \middle| C_{1;j}, C_{2;j} \geq 0 \right\}$$

by

$$\begin{aligned} U_j(B_j, I_j) :=& u_j\big(F_{1;j} + \frac{B_j - I_j}{P_1}\big) \\ &+ \beta_j \left(p_j u_j\big(F_{2;j}(G_j, I_j) - \frac{B_j(r_\# + 1)}{P_1}\big) + (1 - p_j) u_j\Big(\max\{F_{2;j}(NG_j, I_j) - \frac{B_j(r_\# + 1)}{P_1}, 0\}\Big) \right) \end{aligned} \tag{12a}$$

then

$$\forall C_{1;j}, C_{2;j} \geq 0, \quad \mathbb{E}\Big[u_j(C_{1;j}) + \beta_j u_j(C_{2;j})\Big] = U_j(B_j, I_j) \tag{12b}$$

$$\forall (B_j, I_j) \in W_j, \quad \mathbb{E}\Big[u_j(C_{1;j}) + \beta_j u_j(C_{2;j})\Big] = U_j(B_j, I_j) \tag{12c}$$

Therefore, the maximisation problem in Setting 7 is solved by $(C_{1;j}, C_{2;j})$ *if and only if their corresponding* (B_j, I_j) *solves the maximisation problem of*

$$\max_{(B_j, I_j) \in W_j} U_j(B_j, I_j) \tag{12d}$$

Lemma 1 transfers the complication of Setting 7 into a two-dimensional optimisation problem.[38] But U_j is not differentiable everywhere[39], so strict concavity is not guaranteed. Thus the common techniques that economists use, e.g. Lagrange Multiplier methods, are not suitable for this problem. Therefore, the fundamental analysis shall be started as below.

Note on lower scripts: for the rest of this subsection, I drop all j and $\#$ for notational simplicity, e.g. I write B instead of B_j, I instead of I_j, C_1 instead of $C_{1;j}$, and r instead of $r_\#$.

[38] Proof is in subsection 13.1.

[39] E.g. notice the max and the $r_\#$ in Equation 12a — U_j being not differentiable at the points when $\max\{..., 0\}$ and $r_\#$ change values.

Theorem 2. *Existence of Maximiser.*
Consider the following three statements:

(a) $\exists\ \widehat{I} \in \mathbb{R}_{\geq 0}$ *such that* $F_2(G, \widehat{I})P_1(r+1)^{-1} + P_1F_1 - \widehat{I} = 0$.

(b) W *is compact.*

(c) U *has at least one maximiser.*

Then $(a) \iff (b) \implies (c)$.

A mathematical intuition about $(a) \iff (b)$ is reflected in Figure 11. W is bounded if and only if the green line inclines more upward to meet the red line, so that the shaded area ends. [40]

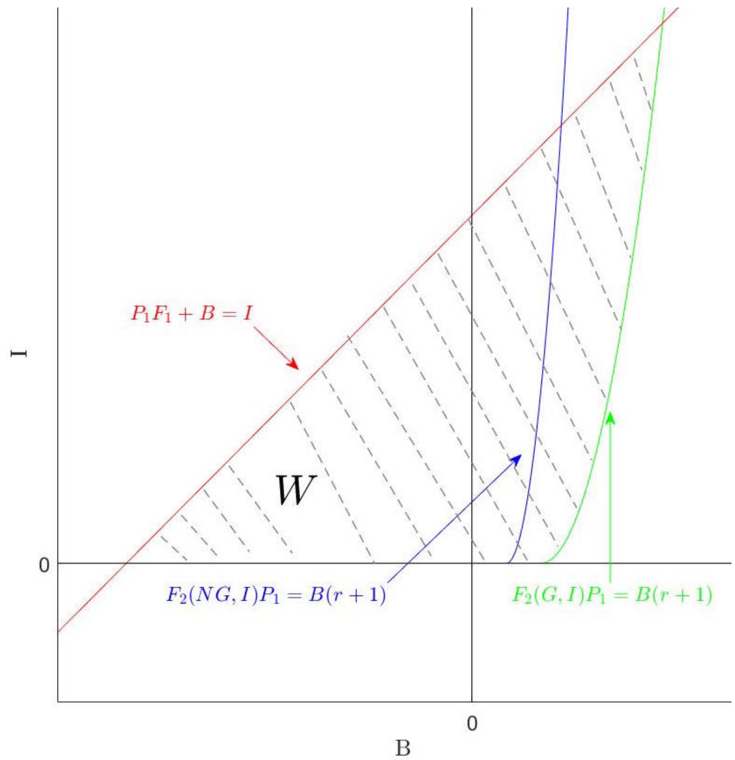

Figure 11: W is the shaded area, including the boundaries surrounding the area.

Theorem 2 gives a sufficient condition for the existence of maximiser by $(b) \implies (c)$, with an equivalence statement ($(a) \iff (b)$) that makes the condition being more practical. Having realised the existence, I now create a recipe by dividing W into three areas, so that unique maximiser can be guaranteed, as shown overleaf.

[40] Proof of Theorem 2 is in subsection 13.2.

Corollary 3. *Recipe for finding Maximisers of U.*
Suppose W is compact. Define

$$V_1 := \left\{ (B,I) \in \mathbb{R}^2 \middle| P_1F_1 + B \geq I \text{ and } I \geq 0 \text{ and } B \leq 0 \right\}$$
$$V_2 := \left\{ (B,I) \in \mathbb{R}^2 \middle| P_1F_1 + B \geq I \text{ and } F_2(NG,I)P_1(r+1)^{-1} \geq B \text{ and } I \geq 0 \text{ and } B \geq 0 \right\}$$
$$V_3 := \left\{ (B,I) \in \mathbb{R}^2 \middle| P_1F_1 + B \geq I \text{ and } F_2(G,I)P_1(r+1)^{-1} \geq B \geq F_2(NG,I)P_1(r+1)^{-1} \text{ and } I \geq 0 \right\}$$

then

(a)
$$\bigcup_{k=1}^{3} V_k = W$$

(b) V_1 and V_2 are convex, and U is strictly concave on V_1 and V_2.

(c) $\forall k \in \{1,2\}$, $U|_{V_k}$ has exactly one maximiser. $U|_{V_3^\circ}$ has at most one maximiser.

(d) $\forall k \in \{1,2,3\}$, $x \in V_k^\circ$ is an interior maximiser for $U|_{V_k}$ if and only if $\nabla U(x) = 0$.

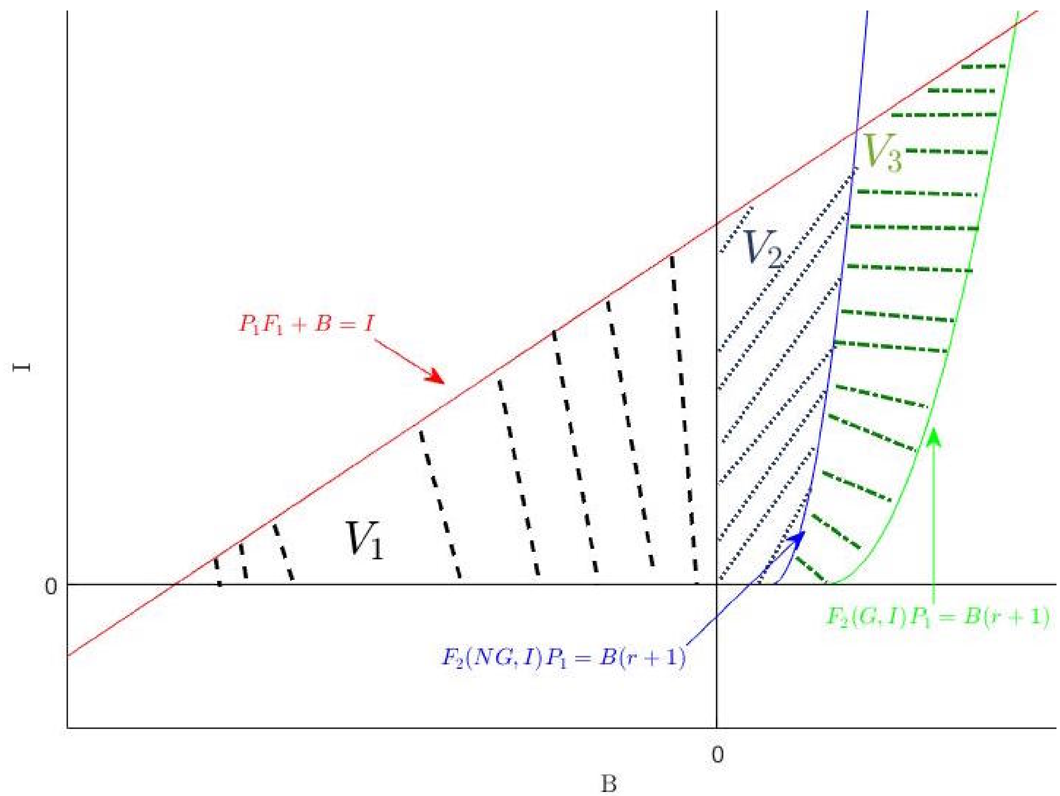

Figure 12: V_1, V_2, V_3 are the shaded area respectively, including the boundaries surrounding their own area.

An economic intuition from Corollary 3[41] and Figure 12 about the $\{V_k\}_{k=1}^3$ is as follows. V_1 is the area where farmer deposits, V_2 is the area where farmer borrows, but in a small amount such that full repayment can be made in all weather, and V_3 is the area where farmer borrows so much such that full repayment can only be made in good weather.

Corollary 3(a) ensures Figure 12 to be a "correct graph" — the union of the three sets equals to W. Then (b) provides a mathematical foundation for (c) and (d). From (c), there is one unique maximiser for each of V_1 and V_2, and (d) gives the exact requirement for an interior maximiser. (d) means, if the interior maximisers in $\{V_k\}_{k=1}^3$ exist, then they are unique in their set, and there would be no need to check the boundaries. Otherwise, if the interior maximiser does not exist, then (c) ensures that, maximisers of V_1 and V_2 must lie on boundaries. The following flowchart summarises the recipe.

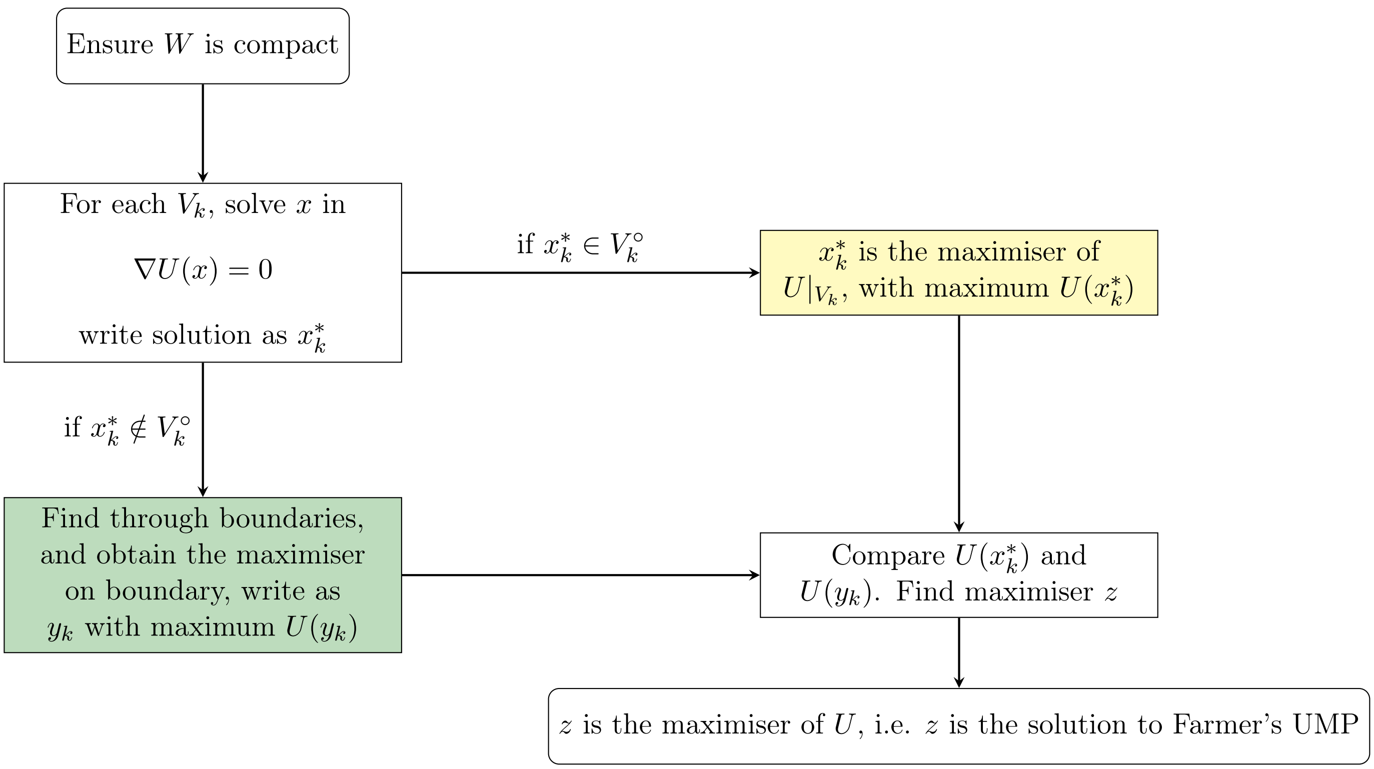

Figure 13: General flowchart for finding solution to Farmer's UMP

In Figure 13, the green shaded box means maximiser shall be found by boundaries, and the yellow shaded box means maximiser is the one in interior satisfying $\nabla U(x) = 0$. Now, I write the boundaries overleaf.

[41] Proof is in subsection 13.3.

Set	Boundary it belongs to	Mathematical Expression	Corresponding line in Figure 12
S_1	Boundary of V_1, V_2, and V_3	$\left\{(B,I)\in\mathbb{R}^2 \,\middle\vert\, P_1F_1+B=I \text{ and } \widehat{I}\geq I\geq 0\right\}$	Red line
S_2	Boundary of V_1, V_2, and V_3	$\left\{(B,0)\in\mathbb{R}^2 \,\middle\vert\, P_1F_2(G,0)(r+1)^{-1}\geq B\geq -P_1F_1\right\}$	Horizontal black line
S_3	Boundary of V_1 and V_2	$\left\{(0,I)\in\mathbb{R}^2 \,\middle\vert\, P_1F_1+B\geq I\geq 0\right\}$	Vertical black line
S_4	Boundary of V_2 and V_3	$\left\{(B,I)\in\mathbb{R}^2 \,\middle\vert\, P_1F_1+B\geq I\geq 0 \text{ and } P_1F_2(NG,I)(r+1)^{-1}=B\right\}$	Blue line
S_5	Boundary of V_3	$\left\{(B,I)\in\mathbb{R}^2 \,\middle\vert\, P_1F_1+B\geq I\geq 0 \text{ and } P_1F_2(G,I)(r+1)^{-1}=B\right\}$	Green line

Table 2: Boundaries
Note: $\widehat{I}\in\mathbb{R}_{\geq 0}$ satisfies $F_2(G,\widehat{I})P_1(r+1)^{-1}+P_1F_1-\widehat{I}=0$

The above finishes the general solution of Farmer's UMP as described in Setting 7.

Notice in Setting 8 (or Figure 9) that, a farmer whose credit application got rejected needs to solve its UMP again with $B\leq 0$. In an economic sense, an "alternative plan on investment and savings" due to credit rationing. Realise that,[42] the new UMP is equivalent to the optimisation problem over V_1. Finally, I conclude the following flowchart.

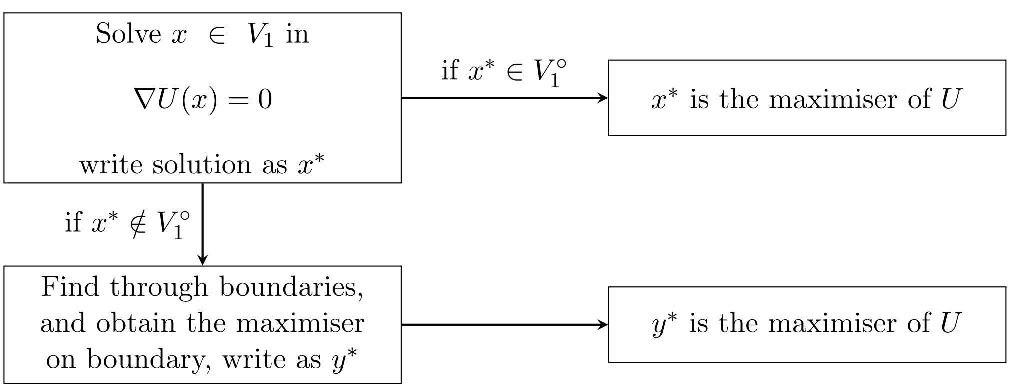

Figure 14: Flowchart for finding solution to Farmer's UMP with $B\leq 0$

[42] By referring back to Lemma 1 and Figure 12.

7.1.2 Comment on Linear Functions

The linear function is commonly used as a start-off point for investigation. But linear function is not strictly concave, which does not satisfy the settings, so I do not use linear functions. Moreover, there are two intuitive explanations.

Mathematically, if one goes through the proofs in section 13, one can realise the reliance of strict-concavity in the proofs, and if strict-concavity not being guaranteed, then the proofs fail. Another argument can be made about the Hessian matrix (i.e. second Frechet derivative) being zero due to linearity.

Economically, consider one linear utility function. Because of the linearity, I can say w.l.o.g., consuming 2 units today and 1 unit tomorrow are equivalent everywhere. Now, suppose 10 units today plus 10 units tomorrow is the optimal combination, and 8 units today plus 11 units tomorrow is feasible. Then, due to the said linearity, 8 units today plus 11 units tomorrow is optimal, too. This goes on and the uniqueness of solution is not guaranteed. The same argument applies to the linear production function.

Therefore I do not consider linear functions as part of the functional specifications.

7.1.3 Comment on Log Utility Function with Power Production Function

Another common function to use is the log functions for utility, and power function (i.e. Cobb–Douglas function) for production. So, consider the following definitions:

Definition 1. Log Utility Function.

$$u(x) = \log(x+1) \tag{13}$$

By verifying the conditions in Setting 4, I conclude Definition 1 fits my setting.

Definition 2. Power Production Function.

$$F_1 = \alpha \tag{14a}$$

$$F_2(G, I) = \alpha + I^{\theta} \tag{14b}$$

$$F_2(NG, I) = \delta(\alpha + I^{\theta}) \tag{14c}$$

where $\delta \in (0,1), \theta \in (0,1), \alpha > 0$

By verifying the conditions in Setting 6, I conclude Definition 2 fits my setting.

Definition 3. Price Normalisation.

$$P_1 = 1 \tag{15}$$

Definition 3 is a common technique to simplify the mathematics in an economic model. Now, Theorem 2(a) is satisfied by the production function in Definition 2, thus W is compact, and I can proceed to find maximisers. For V_1 and V_2, solving $\nabla U(x, y) = (0, 0)$ is same as solving the following system

$$\frac{1}{1+\alpha+x-y} = \beta(r+1)\Big(\frac{p}{1+\alpha-x(r+1)+y^\theta} + \frac{1-p}{1+\delta(\alpha+y^\theta)-x(r+1)}\Big) \tag{16a}$$
$$\frac{1}{1+\alpha+x-y} = \beta\theta y^{\theta-1}\Big(\frac{p}{1+\alpha-x(r+1)+y^\theta} + \frac{\delta(1-p)}{1+\delta(\alpha+y^\theta)-x(r+1)}\Big) \tag{16b}$$

To solve Equation 16, I need to solve y as a function of x from Equation 16a, then use the result to solve x in Equation 16b. Now, when solving Equation 16a, I must multiply the denominators on both sides to eliminate the fraction, then make substitutions such that the equation becomes a polynomial. Then[43], Fundamental Theorem of Algebra guarantees the existence of solution(s) in complex space. But, how can I ensure the solution to be in real space? By Galois Theory, further assumptions may be[44] necessary to ensure real roots, and ensure them to be in a suitable form such that I can use them to solve x in Equation 16b. Exactly the same issue happens even if I have a closed-form solution of y to x, when solving Equation 16b.

Besides the above argument, I also practically solved Equation 16a under $\theta = 0.5$. But the solution was page-long, thus not presented here.

The conclusion is, log utility function, despite being used frequently in examples of economics papers, is not suitable to solve for closed-form solution in my paper.

7.1.4 Comment on Power Utility Function with Power Production Function

A generalisation of log utility function is called isoelastic utility function, i.e. power utility function, which takes the form

$$u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$$

where $\gamma > 0$ and $\gamma \neq 1$. But, when solving for $\nabla U(x, y) = (0, 0)$ in V_1 and V_2, the system yields more complicated format than Equation 16, and by the same argument in subsubsection 7.1.3, it is hard to solve explicitly for closed-form solution.

Thus, I do not consider power utility function in this paper.

[43] These arguments use the results from Galois Theory. subsection 5.4 can be referred for further information.

[44] Not must be, as the study of Galois Theory is a contemporaneous mathematics research topic, with more and more results being found.

7.1.5 Solution for Quadratic Utility Function with a specific Power Production Function

What to be learnt from subsubsection 7.1.3 and subsubsection 7.1.4 is to avoid having fractions[45] in the system, and to avoid irrational powers on y.

The fractions in Equation 16 are due to $u'(x) = \frac{1}{x+1}$. So, what if $u'(x)$ being linear, thus fractions being avoided?

Similarly, y^θ is caused directly by the power production function. So, what if I specify a "good" θ so that solution can be found easily in the y^θ polynomial?

The answer to the first question is by setting a quadratic utility function, as the quadratic function is the anti-derivative of linear function; secondly, set $\theta := 0.5$ and make the "shape" between $F_2(G, \cdot)$ and $F_2(NG, \cdot)$ the same so that polynomials regarding y^θ can be solved easily. The mathematical moderation is made as follows:

Definition 4. A Specific Production Function.

$$F_1 = \alpha$$
$$F_2(G, I) = \alpha + I^{0.5}$$
$$F_2(NG, I) = \delta\alpha + I^{0.5}$$

where $\delta \in (0, 1), \alpha > 0$

Definition 5. Quadratic Utility Function.

$$u(x) = -x^2 + \eta x$$

By verifying the conditions in Setting 6, I conclude Definition 4 fits my setting on production function. However,

$$u'(x) = -2x + \eta \to -\infty \qquad \text{as} \quad x \to \infty$$

so Definition 5 does not satisfy the requirements by Setting 4.[46] But can I change Setting 4? Realise from the start, i.e. Lemma 1 that only W is concerned, and subsequent theorem and corollary with proofs[47] consider in a closed and bounded set, as the power production function ensures W to be compact. Hence, I moderate Setting 4 by restricting U to a sufficiently large set, so that existing theorems remain valid, whilst quadratic utility function is accepted.

[45] More generally, terms with negative power.
[46] As $u'(x) > 0$ is violated.
[47] See section 13 for further details.

Setting 15. Moderated Definition for Utility Function.
Suppose W is compact. Write $\widehat{I} \in \mathbb{R}_{\geq 0}$ to be the solution to $F_2(G, \widehat{I})P_1(r+1)^{-1} + P_1F_1 - \widehat{I} = 0$. Define $\phi := F_1 + F_2(G, \widehat{I})$. Then set the farmer's utility at a given period t to be modelled by a utility function

$$u : [0, \phi] \to \mathbb{R}, \qquad C_t \mapsto u(C_t) \tag{17a}$$

such that

$$u \in C^2([0, \phi], \mathbb{R}) \tag{17b}$$

$$\forall x \in [0, \phi], \quad u'(x) > 0 \quad \text{and} \quad u''(x) < 0 \tag{17c}$$

Setting 15 moderates Setting 4 by replacing the derivative requirements from $\mathbb{R}_{\geq 0}$ to a smaller set $[0, \phi]$, so that Definition 5 can be accepted as long as $\eta > 2\phi$. Moreover, by the definition of ϕ, it is set to be sufficiently large such that proofs made in section 13 still holds.

In an economic sense, the moderation so far is to set in mathematically, the fact that households only consider the feasible sets of goods, rather than an infinitely large amount which is not feasible.

Now, I observe the solution[48] to $\nabla U(x, y) = (0, 0)$, firstly on V_1 and V_2:[49]

$$y = \frac{1}{4(r+1)^2} \tag{18a}$$

$$u'(\alpha + x - y) = \beta(r+1)\Big(pu'(\alpha + \sqrt{y} - x(r+1)) + (1-p)u'(\delta\alpha + \sqrt{y} - x(r+1))\Big) \tag{18b}$$

Then Equation 18b can be further solved in closed-form:

$$x = -\frac{2\,\alpha - \eta - \frac{2}{(2\,r+2)^2} + \beta(r+1)\left((p-1)\left(2\alpha\delta - \eta + \frac{1}{r+1}\right) - p\left(2\,\alpha - \eta + \frac{1}{r+1}\right)\right)}{2\beta(r+1)^2 + 2} \tag{18c}$$

Lastly, observe the solution to $\nabla U(x, y) = (0, 0)$ on V_3:

$$y = \frac{1}{4(r+1)^2} \tag{19a}$$

$$x = \frac{\eta - 2\alpha + \frac{2}{(2\,r+2)^2} + \beta p\,(r+1)\left(2\alpha - \eta + \frac{1}{r+1}\right)}{2p\beta(r+1)^2 + 2} \tag{19b}$$

[48] Price normalisation, i.e. Definition 3 is inherited.
[49] One may actually find two more roots. But notice they are outside of the set — a straightforward reason is, those two roots make $u'(x) \leq 0$ thus conflicts with the given settings.

7.1.6 Summary about Functional Specifications

The general solution needs $\nabla U(x, y) = (0, 0)$ to be solved clearly, whereas log or power utility functions give unclear solutions. Linear functions are too simplified such that the troublesome uniqueness issues arise. At last, quadratic utility function combined with a specific power production function do the job, with slight moderations in the definition of utility function, which does not affect the previous proofs. Therefore, for the rest of my paper, I take subsubsection 7.1.5 as a specification for numerical examples.

7.2 $ComB$'s UMP

The other half of the decision problems in my model, as drawn in Figure 9, is the side from $ComB$.

Definition 6. Outcome Space and Probability Function.
The outcome space and probability function are[50]

$$\Omega := \bigcup \left\{ \{X_1, ...X_n\} \middle| X_j \in \{G_j, NG_j\} \ \forall j \right\}$$

$$\mathbb{P} : \Omega \to [0, 1], \quad \{X_1, ...X_n\} \mapsto \prod_{j=1}^{n} (p_j)^{\mathbb{1}[X_j = G_j]} (1 - p_j)^{\mathbb{1}[X_j = NG_j]}$$

Definition 6 enables me to set the calculation of expected utility as below.

Setting 16. $ComB$'s Expectation.
Write $\Omega = \{\omega_k\}_{k=1}^{2^n}$, then

$$\mathbb{E}[u_{ComB}(L_1(2) - L_1(t))] = \sum_{k=1}^{2^n} (u_{ComB}(L_1(2) - L_1(t)) | \omega_k) \mathbb{P}(\omega_k)$$

Similar to Setting 14, Setting 16 is a standard realisation given I set up the outcome space and probability functions. Now, the following lemma starts the solution to $ComB$'s UMP.

Lemma 4. *Consider two allocations of assets that satisfy Equation 9, call them S_1 and S_2. Let S_1 and S_2 to be exactly the same on all the lendings, but S_1 allocates a higher amount of assets to Cash than S_2. Write the expected utility under S_1 as E_1, and under S_2 as E_2.*

(a) Given $i_c > 0$, then $E_1 > E_2$.

(b) Given $i_c < 0$, then $E_1 < E_2$.

[50] Equivalently, one can write

$$\Omega := \bigcup_{X_n \in \{G_n, NG_n\}} ... \bigcup_{X_1 \in \{G_1, NG_1\}} \{\{X_1, ..., X_n\}\}$$

Lemma 4 has a strong intuitive meaning. Recall Equation 2a and Equation 2b. Since interest rate on bond before the NIR is higher than zero, any extra holdings on bonds, ceteris paribus, should yield higher profits due to higher interest receivables. Similarly, when interest rate on bond is negative after NIR, any extra holdings on bonds, ceteris paribus, lead to higher interest payables, thus yield lower profits. Since utility function is strictly increasing, [51] lower profit means lower expected utility and vice versa, which leads to the result. Since the proof is just the above accounting, it is not provided here given the clear logic.

Now, because it is always preferable, as long as Equation 9 being satisfied, to have more bonds when interest rate is positive and vice versa, I have the following result.

Theorem 5. *Equation 8 is solved by* $\left\{\{B_j^*\}_{j=1}^n, A_0^*(t), A_1^*(t)\right\}$ *if and only if the following problem is solved by* $\left\{\{B_j^*\}_{j=1}^n, A_0^*(t), A_1^*(t)\right\}$ *:*

$$\max_{\{B_j^*\}_{j=1}^n} \mathbb{E}[u_{ComB}(L_1(2) - L_1(t))] \tag{20}$$

where

$$A_0^*(0) = \lambda_0(L_0(0) + L_1(0)) \tag{21a}$$

$$A_1^*(0) \geq \lambda_1(L_0(0) + L_1(0)) \tag{21b}$$

$$A_0^*(1) \geq \lambda_0(L_0(1) + L_1(1)) \tag{21c}$$

$$A_1^*(1) = \lambda_1(L_0(1) + L_1(1)) \tag{21d}$$

Theorem 5[52] gives a simplification to Setting 12. Equation 20 combined with Equation 21a and Equation 21b imply, when interest rate is positive, solution to $ComB$'s UMP must have cash at the minimum level. Similarly, Equation 20 combined with Equation 21c and Equation 21d imply, when interest rate is negative, $ComB$ must have bonds at the minimum level to solve its UMP. Therefore, the only choice variables being left is $\{B_j^*\}_{j=1}^n$, which is to determine whether to grant the credit or not.

The above finishes the general solution for $ComB$'s UMP. Noticeably, because of the settings, $ComB$ is less complicated than farmers, in terms of finding solutions toward the result. So, most of the common functions for utilities can be used, including $\log(x)$ and $-e^{-x}$.

[51] Note: an implication from $u'_{ComB}(x) > 0$ is that u_{ComB} being a strictly increasing function.

[52] Proof is in subsection 13.4.

8 Results: Numerical Examples

8.1 Introduction

From Farmer's UMP, no general result is found. Since the lack of rules on parameters, the interior maximiser is ambiguous[53], thus the result is unclear without further specification. Similarly, for $ComB$, different parameters may yield different results.

Here, I select some of the numerical specifications to present.

8.2 Risk-free Lendings

8.2.1 Farmer

Consider $n = 1$, i.e. one farmer, with the following scalar values:

$$\alpha = 1, \quad \eta = 100, \quad \beta = 1, \quad \delta = 0.9, \quad p = 0.9, \tag{22a}$$

$$\pi = 0.01, \quad r_b(0) = r_b(1) = 0.01, \quad r_d(0) = r_d(1) = -0.01 \tag{22b}$$

Numbers in Equation 22a are settings for farmer's production and utility functions. Noticeably, η is indeed large enough such that Setting 15 is satisfied. Other values satisfy the requirements as set in section 6, too. Numbers in Equation 22b are settings for the financial market, and one can calculate the nominal interest rates as $i_b(0) = i_b(1) = 0.0201$ and $i_d(0) = i_d(1) = 0$, so Preliminary Assumption 2 is satisfied. Now, I follow the steps outlined in Figure 13 and Figure 14 to solve the Farmer's UMP.

W is indeed compact[54], then I solve[55] $\nabla U(x) = 0$ and obtain the following table.

j	x_j^*	In V_j° ?
1	(0.624,0.255)	No
2	(0.121,0.245)	Yes
3	(2.687,0.245)	No

Table 3: Interior Solutions.
Note on approximation: MATLAB 3rd approx.

From Table 3, I know $x_2^* \in V_2^\circ$, thus x_2^* is the maximiser of $U|_{V_2}$, and the next steps in Figure 13 is to find boundary maximisers for V_1 and V_3. [56]

[53] It exists in $\mathbb{R}^2$, but may not be inside the relevant set.

[54] One way is to verify Theorem 2(a) by straightforward solving. Another (elegant) way is to argue by derivative and limits of the functional induced by the equation in Theorem 2(a).

[55] i.e. I plug the values into Equation 18 and Equation 19.

[56] Of course, boundaries of V_2 are exempted because x_2^* is the maximiser of $U|_{V_2}$. i.e. S_3 and S_4 are exempted. But, I still calculate S_2 because it is the boundary of V_1, which will be used later in the process of Figure 14.

Set (refers to Table 2)	Maximiser	Maximum	Geometric Comment (refers to Figure 12)
$S_1 \cap V_1$	(-0.745,0.255)	218.268	n.a.
$S_2 \cap V_1$	(0,0)	197.019	int.pt. between horizontal black and vertical black lines
S_3	(0,0.243)	221.222	n.a.
$S_1 \cap V_3$	(2.828,3.828)	8.991	int.pt. between Blue line and Red line
$S_2 \cap V_3$	(0.990,0)	195.049	int.pt. between horizontal black line and Green line
S_5	(1.480,0.245)	218.521	n.a.

Table 4: Boundary Maximisers and Maxima.
Note on approximation: MATLAB 3rd approx.

[57] Now, having finished the finding of boundary maximisers, I move to the last step of Figure 13, which is to compare $U(x_2^*)$ against the maxima found in Table 4. Since $U(x_2^*) \approx 221.251$, which is greater than all the boundary maxima, I conclude x_2^* to be the solution to the farmer's UMP.

Finally, I follow Figure 14 to find farmer's UMP solution if $B \leq 0$, i.e. when credit is rationed. By Table 3, as $x_1 \notin V_1^\circ$, the maximiser must lie on the boundary. Then by Table 4, the maximiser is $(0, 0.243)$, as $221.222 > 218.268 > 197.019$.

To conclude, the farmer applies a credit of[58] £0.121 and invests £0.245 if credit gets granted. In case credit application not being granted, the farmer would deposit £0 and invest £0.243.

8.2.2 *ComB*

Consider the following scalar values:

$$L_1(0) = L_1(1) = 1, \quad \lambda_0 = \lambda_1 = 0.2, \quad i_c(0) = 0.01 \tag{23}$$

Equation 23 sets some numerical values, including £1 of equity and 1% central bank interest rate.

From the farmer's UMP, as $x_2^* \in V_2^\circ$, the farmer is able to return the credits for all weathers, i.e. the lendings are risk-free.[59] Thus, if credit is granted, then

$$\forall \omega \in \Omega, \quad A_2(2) = (1 + i_b)B^* = 1.0201 \times 0.121 \approx \text{(4th decimal place) } 0.1234$$

otherwise if credit is not granted, then $\forall \omega \in \Omega, \quad A_2(2) = 0$

[57] Note on techniques in finding boundary maximisers: for each boundary, the restricted function can be represented by a one-dimensional function. Then, typical undergraduate-level techniques for observing the first order derivative are utilised, and further details can be found in books mentioned in subsection 5.4. Since these are standard bookwork of one-dimensional analysis, no further details are provided in my paper.

[58] "£" sign is used as a monetary measurement.

[59] $x_2^* \in V_2^\circ$ implies (write x_2^* in coordinate form (B, I)) $F_2(NG, I)P_2 > B(i_b + 1)$, which means production greater than payables even when bad weather happens.

Now, by Theorem 5, I complete[60] the balance sheet at period 2 and thus calculate the implied profits when $i_c > 0$ is as follows:

Item in Period 2 as $i_c = 0.01$	If Credit is not granted	If Credit is granted
$A_0(2)$	0.2	0.2
$A_1(2)$	0.808	0.6858
$A_2(2)$	0	0.1234
$L_0(2)$	0	0
$L_1(2) - L_1(0)$ (Implied profits)	0.008	0.0092

Table 5: Calculation towards the implied profits.
Note: the full settings of the items in Balance sheet are in Figure 10, and the calculation of profits follows from Setting 10.
Note on approximation: MATLAB 4th approx.

Similarly, the implied profits when $i_c < 0$ is as follows:

Item in Period 2 when $i_c < 0$	If Credit is not granted	If Credit is granted
$A_0(2)$	0.8	0.679
$A_1(2)$	$0.2(1+i_c)$	$0.2(1+i_c)$
$A_2(2)$	0	0.1234
$L_0(2)$	0	0
$L_1(2) - L_1(1)$ (Implied profits)	$0.2i_c$	$0.0024+0.2i_c$

Table 6: Calculation towards the implied profits.
Note on approximation: MATLAB 4th approx.

Now, for the case when $i_c = 0.01$, as $0.0092 > 0.008$, I conclude the $ComB$ would grant the credit for any utility functions that satisfy Setting 11.[61] Similarly, when $i_c < 0$, as $0.0024 + 0.2i_c > 0.2i_c$, I conclude the $ComB$ would grant the credit for any utility function, too.

Finally, consider back to Conjecture 1 made in Part I. The conjecture fails here, as lendings do not decrease. Moreover, there are no change in consumption[62] and investment since credits are not rationed.

8.3 Risky lendings

In subsection 8.2, since farmer is borrowing without risks, there are no risks taken by $ComB$. So, what if the farmer was borrowing with risks thus $ComB$ needed to consider credit default risks? I now present another example to illustrate this. [63]

[60] Since Theorem 5 sets out the calculation method, the remaining steps are just to plug in the values, which is purely an accounting exercise. Hence these steps are skipped.
[61] Because $u(x)$ is a strictly increasing function, $u(0.0092) > u(0.008)$.
[62] One can also calculate the consumption using the numbers back to subsubsection 8.2.1, and realise no change in consumption.
[63] Note: no middle steps are shown here, as the steps are the same as subsection 8.2.

8.3.1 Farmer

Inherit the scalar values from Equation 22, but with decreased β, δ, and increased p:

$$\beta = 0.98. \qquad \delta = 0.2, \qquad p = 0.99$$

Intuitively, a decreased δ means the production under bad weather is now less than previous, given the same investment. An increased p means, good weather is more likely than previous. The latter intuitively motivates the farmer to borrow more — there is more likelihood of good weather, thus less likelihood of bad weather that causes potential credit default, so the expected "bad life" due to credit default is valued less because of the lower likelihood, thus more borrowings are encouraged. Likewise, a decrease in β means consumption in period 1 is more "valuable" than period 2, thus more period 1 consumptions and borrowings are preferable, ceteris paribus compared to a higher β.

The followings are a summary of maximisers by sets:

Set	Maximiser	Interior or Boundary	Maximum
V_1	(0,0.233)	Boundary	218.495
V_2	(0.614,0.245)	Interior	219.233
V_3	(0.862,0.245)	Interior	219.282

Table 7: Maximisers by sets.
Note on approximation: MATLAB 3rd approx.

As shown in Table 7, $(0.862, 0.245) \in V_3^\circ$ is the maximiser, meaning the farmer would apply for a credit that involves default risks. Also, in case that credit is not granted, $(0, 0.233)$ would be the alternative plan. That is, the farmer applies £0.862 of borrowings first, if the credit is rejected, then £0 would be the deposit. A summary is as follows: [64]

Item	If Credit is not granted	If Credit is granted
C_1	0.767	1.617
I	0.233	0.245

Table 8: Consumption and Investment in Period 1.
Note on approximation: MATLAB 3rd approx.

[64] I calculate the consumptions by Equation 5b, and investment is a reiteration of the data in Table 7.
Note: C_2 is not considered here due to the two natures from settings: first, C_2 depends on the weather, so cannot be summarised as one number; second, period 2 is the last period of time, meaning (one can also see this from the settings) all the remaining money are spent, which also blurs the nature of consumption behaviour. This can be dealt with, later in subsection 9.1, once more dynamics being introduced.

8.3.2 *ComB*

Consider the following scalar values:

$$L_1(0) = L_1(1) = 2, \quad \lambda_0 = \lambda_1 = 0.2, \quad i_c(0) = 0.01 \tag{24}$$

Equation 24, similar to Equation 23, sets some numerical values, including £2 of equity. By the same process as subsubsection 8.2.2, I calculate the implied profits as follows:

	If Credit is not granted	If Credit is granted and Good Weather next period	If Credit is granted and Bad Weather next period
Implied profit as $i_c = 0.01$	0.0160	0.0247	-0.1596
Implied profit when $i_c < 0$	$0.4i_c$	$0.0173+0.4i_c$	$-0.1669+0.4i_c$

Table 9: Implied profits under different credit decision.
Note on approximation: MATLAB 4th approx.

So, the expected utilities in each case are:[65]

	If Credit is not granted	If Credit is granted
EU as $i_c = 0.01$	$u_{ComB}(0.0160)$	$0.99u_{ComB}(0.0247) + 0.01u_{ComB}(-0.1596)$
EU when $i_c < 0$	$u_{ComB}(0.4i_c)$	$0.99u_{ComB}(0.0173 + 0.4i_c) + 0.01u_{ComB}(-0.1669 + 0.4i_c)$

Table 10: EU under different circumstances.
Note on approximation: MATLAB 4th approx.

From Table 10, for $i_c = 0.01$, I know the credit is granted if EU of not granted is lower than EU of granted, which is equivalent to

$$u_{ComB}(0.0160) < 0.99u_{ComB}(0.0247) + 0.01u_{ComB}(-0.1596) \tag{25a}$$

Similarly, when $i_c < 0$, if the following holds,

$$u_{ComB}(0.4i_c) < 0.99u_{ComB}(0.0173 + 0.4i_c) + 0.01u_{ComB}(-0.1669 + 0.4i_c) \tag{25b}$$

then credit is granted; and if

$$u_{ComB}(0.4i_c) > 0.99u_{ComB}(0.0173 + 0.4i_c) + 0.01u_{ComB}(-0.1669 + 0.4i_c) \tag{25c}$$

then credit is not granted .

[65] Write EU as a shorthand to $\mathbb{E}[u_{ComB}(L_1(2) - L_1(t))]$ for notational simplicity.

Now, I deduce the followings:

Theorem 6. *There are uncountably*[66] *many utility functions for $ComB$ that make the credit to be granted when $i_c = 0.01$, but credit not to be granted when $i_c = -0.01$. Whilst there are also uncountably many utility functions for $ComB$ that make the credit to be granted both as $i_c = 0.01$ and as $i_c = -0.01$.*

The proof is in subsection 13.5, and the plots below show one example from each side.

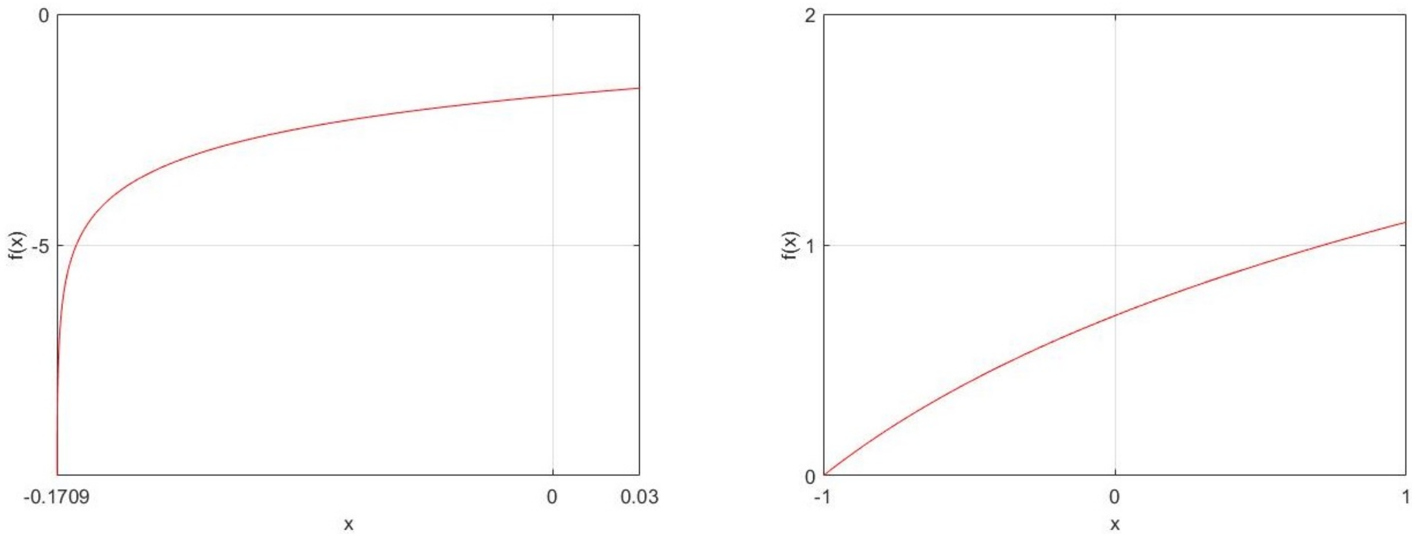

Figure 15: The left function satisfies Equation 25a and Equation 25c, whereas the right function satisfies Equation 25a and Equation 25b.

The function at the left part of Figure 15 leads the $ComB$ to grant the credit before the rate cut, and not to after the rate cut, whereas the function at the right part of Figure 15 leads the $ComB$ to grant the credit both before and after the rate cut.

A mathematical intuition is, the left function declines quickly (from right to left) such that $-0.1669 + 0.4i_c$ got assigned an extremely low value, so Equation 25c holds, whereas the right function is not deepening quickly, so Equation 25b holds.

An economic intuition is, the left function represents more risk aversive preference than the right one. That is, the bank whose utility function being represented by the left function has stronger preference for safer portfolio, compared to the one whose utility function being represented by the right function.

By Table 8, if the credit is granted before NIR and not granted after NIR, then consumption and investment in period 1 decrease due to NIR. But if the credit is being granted both before and after the rate cut, then no change in consumption and investment would occur.

Lastly, looking back to Conjecture 1, it is partially reflected here. By Theorem 6, on the one hand, there are infinitely many utility functions that would make $ComB$ to cease the existing lendings, then consumption and investment decrease because of the rationed credit. On the other hand, some other utility functions

[66] Uncountable is a stronger type of infinite. See textbooks in subsection 5.4 for further details.

may make $ComB$ to carry on the lendings that it granted previously, so no change in consumption and investment.

8.4 Summary

Despite no general result was found, given some sets of parameters, an interesting summary can be drawn.

Firstly, from subsection 8.2, even if one bank faces a decrease in profit due to NIR, the bank may still carry on its lendings as usual, given the lendings are not risky.

Secondly, from subsection 8.3, a combination of a drop in profit and risky portfolio still give ambiguous result — more risk aversive banks may opt to cease the lendings, whilst the others would not.

Lastly, back to the topic question, the examples above have shown an uncertain answer. But, from subsection 8.3, there are infinitely many chances for NIR to cease commercial bank's lendings, thus decrease consumption and investment. This phenomenon shall motivate policymakers and scholars to consider the possibility of the negative impact on consumption and investment, due to a rationed credit brought by NIR.

Part III

Extension, Evaluation, and Conclusion

Overview: section 9 suggests potential extensions to the model in Part II. Then, section 10 gives an evaluation on the entire paper, followed by section 11 that concludes this paper.

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9 Extension to The Model

9.1 Goods Market Equilibrium — An outline for extension

In the previous analysis, prices are set exogenously for mathematical simplicity. This holds in partial equilibrium, because part of the economy may not be large enough to affect the price. To make the result more robust and convincing, an extension to the goods market equilibrium shall be included.

The concept of goods market equilibrium is to endogenise prices so that total consumption is equal to total production. Thus I introduce the following settings.

Setting 17. State Space and Goods Market Clearing.

(a) Define the state space in period 2 as

$$S = \{s_1, ..., s_K\} \tag{26a}$$

where each $s_k \in S$ can be written uniquely as $s_k = \{X_1, ..., X_n\}$ where $X_j \in \{G_j, NG_j\} \quad \forall j \in \{1, ..., n\}$.

Call every element $s \in S$ a state.

(b) Write $C_{s,j}$ as the consumption of agent j in case state s happens. For every state $s \in S$, write the price of banana as P_s per unit. Define the Market Clearing condition to be the prices $P_1, \{P_{s_k}\}_{k=1}^{K}$ such that:[67] (i) Market in period 1 clears:

$$\sum_{j=1}^{n} C_{1,j} = \sum_{j=1}^{n} (F_{1,j} - \frac{I_j}{P_1}) \tag{26b}$$

(ii) Market in every state clears:

$$\forall s \in S, \quad \sum_{j=1}^{n} C_{s,j} = \sum_{j=1}^{n} F_{2,j}(X_j, I_j) \tag{26c}$$

[67] Note: consumption and production are related to prices, through each farmer's UMP.

Equation 26a is a mathematical representation of the state space. Equation 26b sets in the market clearing condition for period 1 — goods consumed shall be equal to the total production with investment[68] taken out. Similarly, Equation 26c sets the condition for every state in period 2.

Two notes can be taken from the market clearings. Firstly, the introduction of state space makes my model to be stochastic. This is because, when farmers solve their UMP, they are facing different prices under different states. Thus expectations amongst stochastic states play a role. This leads to the second note — awareness of different prices due to different states may result in disequilibrium[69], then an extension to a few more periods shall be done to clear such disequilibrium, which adds an elementary dynamic to the model.[70]

Therefore, this subsection draws not only an extension to a sole goods market clearing, but also an underlining setting to convert my model to a DSGE structure, which is a current trend of monetary models.

9.2 Other extensions

To my model itself, there are a number of elements not changed in the examples in section 8, e.g. i_b, i_d. So, a straightforward extension is to show more examples and draw more general conclusions.

My model can be extended further, from the macroeconomic side to include sticky prices; and from the microeconomic side to make changes in credit network, e.g. include a number of different types of banks.

A potential project using an econometrics method to compare Japanese bank's earnings and lending-structures pre- and post-NIR is also called for, so that Conjecture 1 can be verified by empirical data. My model can act as a theoretical support to such an econometrics project.

In summary, there are numerous extensions that can be made to my model, including more empirical works via econometrics.

[68] Exchanged into units of banana.

[69] For example, farmer A may expect farmer B to invest, and have an expected price conditional on such assumption, whereas farmer B may be unable to invest due to rationed credit brought by NIR.

[70] Further theoretical concepts can be found in textbooks such as Walsh (2010), Ljungqvist and Sargent (2012), and other papers mentioned in subsection 5.2.

10 Evaluation

10.1 Highlights

A highlight of my paper is the integration of multiple fields. These include microeconomic decisions for commercial banks, macroeconomic models for consumption and investment, and theoretical mathematics to solve optimisation. Moreover, there is no significant reliance on any particular literature, and my paper brings new ideas to the current debate on NIR.

Compared to some macroeconomic literature, in which the economy is characterised by a system of equations, my model sets two microeconomic decision problems for two different participants — farmers decide on borrowings and investment, whereas $ComB$ decides on making credits or not. So, a macroeconomic story about consumption and investment is told through the microeconomic decisions from $ComB$ trying to maximise its utility on profits, and farmers maximising their utilities on consumption. This indeed coheres what I promised in subsection 5.1 — building microfoundations.

During the solution of the model, mathematical knowledge in concave minimisation is both utilised, and extended above the textbook-level. In particular, the V_3 area[71] disables classical results to be used directly. So, I created my own optimisation solution, by considering the topological properties of V_3.

10.2 Structure

The trade-off of both handling the word count and including multiple fields in this paper, is the extensions in each field. For example, when doing numerical results, I could mathematically extend it to a more generic statement, but I rather used words to explain my current results so that economic intuition can be drawn. Proofs may also not be as smooth as I desired to — some steps were skipped due to page and word limits, but I covered the trickiest non-standard steps in the proofs carefully, so that the main logic behind the proof is clear. Also, the macroeconomic model could be set more generally, but more explanations would be required, thus I decided to only include a simpler version.

A critique may be the unnecessary focus on Japan. The central reason for me to do so, is that Japan is unique, and empirical observations are essential to building my model. Reviews in section 3 and section 4 act as a direction to the fields of literature studied in section 5. Moreover, section 3 and section 4 themselves bring Western scholars an overview of the Japanese economic phenomena, which are partly helpful to explain the settings of my model in Part II. For instance, the crucial phenomenon that Japanese commercial banks hold negatively-yielded government bond is explicitly reflected in my model.

[71] As shown in Figure 12 and further during the proofs, it is not convex.

10.3 Presentation

I avoided this paper becoming a maths workbook by providing economic intuition throughout Part II, even for mathematical statements.

Figures are deliberately utilised in my paper — geometric items and flowcharts are helpful for abstract mathematics, and being included within section 7. Plots in Part I also helped to describe the numerical phenomena in Japan.

10.4 Robustness of The Model

One typical concern for economic models is their robustness.

A reasonable critique is, my paper does not discuss price levels. Indeed, the objective of Japanese monetary policy is to reach an inflationary target, rather than boosting consumption and investment (Bank of Japan 2013). To address such critique, prices are endogenised in the extension in subsection 9.1, which provides a future outline to a complete model.

Certainly, one may worry about the functional specifications and the assumptions themselves, but this may be hard to test without econometric practices. Noticeably, parts of the key assumptions come from the empirical observations done in section 4, thus should be of high validity. Another positive side of my model is, no approximation (or linearisation) is used to find closed-form solutions, therefore no concern shall arise on the closed-form solutions.

11 Conclusion

Part I firstly explains the reason to come up with the topic questions, secondly observes economic facts in Japan to come up with assumptions and the conjecture, lastly reviews the key papers in macroeconomics, microeconomics, and mathematics to build and solve my model.

Part II builds intuitive settings in the micro-founded economic model in order to address the macroeconomic topic question. Then, theoretical mathematics solves the model. Numerical results suggest an uncertain answer to the topic question. i.e. there are infinitely many possibilities for a drop in consumption and investment due to NIR, but NIR may also have no negative impact on consumption and investment in some other cases.

Finally, Part III suggests some potential extensions and gives an evaluation to this paper.

Part IV

Appendix

12 Mathematical Definitions and Useful Results

12.1 Remark on notations

Some economics papers use "$\mathbb{R}_+$" to define the set of all non-negative real numbers. However, common mathematics papers use $\mathbb{R}_{\geq 0}$, and so does my paper. Similarly, log is used instead of "ln" to represent natural logarithm.

Consider $a < b$ and $f : [a, b] \to \mathbb{R}$, some books may say $f'(a)$ not exist (especially when we consider Frechet derivative). In my paper, I define[72]

$$f'(a) = \lim_{x \to a^+} f'(x) \qquad \text{and} \qquad f'(b) = \lim_{x \to b^-} f'(x)$$

If both limits exist, and f being differentiable on (a, b), then I say f to be differentiable on $[a, b]$.

Consider two sets A and B. For $k \in \mathbb{N}_{\geq 1}$, the set of k-times continuously differentiable functions $f : A \to B$ is written as $C^k(A, B)$. The set of continuous functions $f : A \to B$ is written as $C^0(A, B)$.

Other notations are standard, and can be found in books mentioned in subsection 5.4.

12.2 Metric Spaces and Functional Maximisations

Throughout my paper, when $\mathbb{R}^n$ is mentioned, $n \in \mathbb{N}_{\geq 1}$ is automatically assumed. I consider the metric space $(\mathbb{R}^n, d)$ where d is the Euclidean distance, and write the open ball $B(x, r) := \{y \in \mathbb{R}^n | d(x, y) < r\}$. For a set A, A° is the interior of A.

Definition 7. Maximisers.
Let X be a non-empty subset of $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$.

(a) Define $x \in X$ to be a global maximiser (abbreviated as maximiser) of f if $\forall y \in X, \quad f(x) \geq f(y)$

(b) Define x to be an interior maximiser of f if x is a maximiser of f and $x \in X^\circ$

(c) Define $x \in X$ to be a local maximiser of f if $\exists r > 0$ such that x is a maximiser of $f|_{B(x,r) \cap X}$

[72] If the limit on the right hand side does not exist, then the left hand side is said to be non-existence.

Theorem 7. *Heine-Borel Theorem and an application (major components from Sutherland (2009, pp.130-132,134), with slight changes in definition).*
Let $A \subset \mathbb{R}^n$.

(a) A is compact $\iff$ A is closed and bounded.

(b) If A is compact, and if $f : A \to \mathbb{R}$ is continuous, then f has at least one maximiser.

Definition 8. First Order Derivative.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Let $x \in \mathbb{R}^n$ and write it in coordinate form as $x = (x_1, ..., x_n)$, define $\nabla f(x) := (\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}(x))$

Theorem 8. *First Order Condition (major components from Beck (2014, pp.16-17,120,148), with slight changes in definition). Let $A \subset \mathbb{R}^n$ be convex and $f : A \to \mathbb{R}$ be concave.*

(a) Suppose $f|_{A^\circ}$ be once continuously differentiable. Then $x \in A^\circ$ is an interior maximiser of f $\iff$ $\nabla f(x) = 0$

(b) If f is strictly concave and x is a local maximiser of f, then x is the unique maximiser of f.

13 Proofs for section 7 and section 8

13.1 Lemma 1

Note: Drop the j and # for notational simplicity.
Proof: by the definition of W, I conclude U to be well-defined. Pick any arbitrary $C_1, C_2 \geq 0$.
By Equation 5c and Equation 10b,

$$C_2(G) = F_2(G, I) - \frac{B(1+i)}{P_2} \tag{27a}$$
$$= F_2(G, I) - \frac{B(1+r)}{P_1} \tag{27b}$$

and

$$C_2(NG)P_2 = \begin{cases} 0 & \text{if } F_2(NG, I)P_2 - B(i+1) \leq 0 \\ F_2(NG, I)P_2 - B(i+1) & \text{otherwise} \end{cases}$$
$$= \max\{F_2(NG, I)P_2 - B(i+1), 0\}$$

Then similar to Equation 27, by dividing P_2 and using Equation 10b, the above can be written as

$$C_2(NG) = \max\{F_2(NG, I) - \frac{B(1+r)}{P_1}, 0\} \tag{28}$$

By Equation 5b,

$$C_1 = F_1 + \frac{B - I}{P_1} \tag{29}$$

Now, use Equation 27b, Equation 28, and Equation 29 to elaborate Equation 11:

$$\begin{aligned}\mathbb{E}\Big[u(C_1) + \beta u(C_2)\Big] =& u(F_1 + \frac{B-I}{P_1}) + \beta\Big(pu(F_2(G,I) - \frac{B(1+r)}{P_1}) + (1-p)u(\max\{F_2(NG,I) - \frac{B(1+r)}{P_1}, 0\})\Big)\\ =& U(B,I)\end{aligned}$$

This proves Equation 12b. Now, pick any arbitrary $(B,I) \in W$, then using the same methods as above, by Equation 11, Equation 27b, Equation 28, and Equation 29, $\mathbb{E}\Big[u(C_1)+\beta u(C_2)\Big] = U(B,I)$. So Equation 12c holds. The remaining proves the maximisation conclusion at the bottom of Lemma 1.

Define the "corresponding element" of (C_1, C_2) as a pair (B,I) such that Equation 5b and Equation 5c are satisfied, and vice versa.

$\Longrightarrow$ direction: suppose (C_1, C_2) solves Setting 7, and (B,I) is the corresponding element. $\forall (x,y) \in W$, write its corresponding element as $a, b \geq 0$. Then by Equation 12c,

$$U(x,y) = \mathbb{E}[u(a) + \beta u(b)] \tag{30a}$$

Since (C_1, C_2) solves Setting 7, so

$$\mathbb{E}[u(a) + \beta u(b)] \leq \mathbb{E}[u(C_1) + \beta u(C_2)] \tag{30b}$$

Since (B,I) is the corresponding element of (C_1, C_2), so

$$U(B,I) = \mathbb{E}[u(C_1) + \beta u(C_2)] \tag{30c}$$

By Equation 30a, Equation 30b, and Equation 30c, I conclude

$$U(x,y) \leq U(B,I), \qquad \forall (x,y) \in W \tag{30d}$$

i.e. (B,I) is the maximiser of U, i.e. (B,I) solves the maximisation problem of

$$\max_{(B,I)\in W} U(B,I)$$

$\Longleftarrow$ direction: suppose (B,I) solves the maximisation problem of

$$\max_{(B,I)\in W} U(B,I)$$

then write (C_1, C_2) as the corresponding element. Then by Equation 12b,

$$U(B, I) = \mathbb{E}[u(C_1) + \beta u(C_2)]$$

$\forall a, b \geq 0$, write (x, y) as its corresponding element. By the similar argument as previous, I can show

$$\mathbb{E}[u(C_1) + \beta u(C_2)] = U(B, I) \geq U(x, y) = \mathbb{E}[u(a) + \beta u(b)]$$

thus conclude (C_1, C_2) solves Setting 7. $\square$

13.2 Theorem 2

$(a) \implies (b)$ direction: suppose $\exists\ \widehat{I} \in \mathbb{R}_{\geq 0}$ such that $F_2(G, \widehat{I})P_1(r+1)^{-1} + P_1F_1 - \widehat{I} = 0$, then define $\widehat{B} := F_2(G, \widehat{I})P_1(r+1)^{-1}$, thus $\widehat{B} = \widehat{I} - P_1F_1$. Then notice

$$(\widehat{B}, \widehat{I}) \in \{(B, I) \in \mathbb{R}^2_{\geq 0} | F_2(G, I)P_1 = B(r+1)\} \cap \{(B, I) \in \mathbb{R}^2_{\geq 0} | F_1P_1 + B = I\}$$

i.e. $(\widehat{B}, \widehat{I})$ is the point of intersection between the green and red line in Figure 11. So W is bounded. [73] Since W is closed by its definition, thus by Theorem 7, W is compact.

$(a) \impliedby (b)$ direction: since W is compact, by Theorem 7, W is bounded, thus $\exists(\widehat{B}, \widehat{I}) \in \mathbb{R}^2_{\geq 0}$ such that[74]

$$(\widehat{B}, \widehat{I}) \in \{(B, I) \in \mathbb{R}^2_{\geq 0} | F_2(G, I)P_1 = B(r+1)\} \cap \{(B, I) \in \mathbb{R}^2_{\geq 0} | F_1P_1 + B = I\}$$

thus $F_2(G, \widehat{I})P_1 = (\widehat{I} - F_1P_1)(r+1)$ which leads to $F_2(G, \widehat{I})P_1(r+1)^{-1} + P_1F_1 - \widehat{I} = 0$.

$(b) \implies (c)$ direction: observe Equation 12a, I conclude U to be continuous on

$$W^* := W \setminus \{(B, I) \in \mathbb{R}^2_{\geq 0} | F_2(NG, I) - \frac{B(r+1)}{P_1} = 0\} \setminus \{(B, I) \in \mathbb{R}^2_{\geq 0} | B = 0, I \leq P_1F_1\} \tag{31}$$

This is because U is a standard composition of continuous functions on W^*. Now, consider the set

$$Q_1 := \{(B, I) \in \mathbb{R}^2_{\geq 0} | F_2(NG, I) - \frac{B(r+1)}{P_1} = 0\}$$

Pick an arbitrary element $\mathbf{x} \in Q_1$ and observe, for $\mathbf{y} \in W$,[75]

$$\lim_{\mathbf{y} \to \mathbf{x}^-} U(\mathbf{y}) - U(\mathbf{x}) = \beta(1-p) \lim_{\mathbf{y} \to \mathbf{x}^-} u(F_2(NG, y_2) - y_1(r+1)P_1^{-1}) - u(0) = 0 \tag{32}$$

[73] One can also prove through more fundamental definition by considering a "big ball" $B(0, K) \supset W$ where $K = 2 \times \max\{1, \widehat{B}, \widehat{I}\}$.

[74] Otherwise W is not bounded due to red line being "higher" than green line in Figure 11. One can prove so by contradiction.

[75] Write the coordinate of $\mathbf{y}$ as (y_1, y_2).

because $F_2(NG, y_2) - y_1(r+1)P_1^{-1} \to 0^+$ as $\mathbf{y} \to \mathbf{x}^-$ and other terms are the same. Similarly, observe

$$\lim_{\mathbf{y}\to\mathbf{x}^+} U(\mathbf{y}) - U(\mathbf{x}) = \beta(1-p) \lim_{\mathbf{y}\to\mathbf{x}^+} u(0) - u(0) = 0 \tag{33}$$

because $F_2(NG, y_2) - y_1(r+1)P_1^{-1} \to 0^-$ as $\mathbf{y} \to \mathbf{x}^+$ and other terms are the same.
From Equation 32 and Equation 33, I conclude

$$\lim_{\mathbf{y}\to\mathbf{x}^+} U(\mathbf{y}) = \lim_{\mathbf{y}\to\mathbf{x}^-} U(\mathbf{y}) = U(\mathbf{x})$$

Thus U is continuous on $Q_1 \cup W^*$. Now, consider the remaining set

$$Q_2 := \{(B, I) \in \mathbb{R}^2_{\geq 0} | B = 0, I \leq P_1 F_1\}$$

Similar to the previous argument, pick an arbitrary element $\mathbf{x} \in Q_2$ and observe, for $\mathbf{y} \in W$,

$$\lim_{\mathbf{y}\to\mathbf{x}^-} U(\mathbf{y}) - U(\mathbf{x}) = \beta p \lim_{\mathbf{y}\to\mathbf{x}^-} u(F_2(G, y_2) - y_1(r_d+1)P_1^{-1}) - u(F_2(G, y_2)) = 0$$
$$\lim_{\mathbf{y}\to\mathbf{x}^+} U(\mathbf{y}) - U(\mathbf{x}) = \beta p \lim_{\mathbf{y}\to\mathbf{x}^+} u(F_2(G, y_2) - y_1(r_b+1)P_1^{-1}) - u(F_2(G, y_2)) = 0$$

because $y_1 \to 0$ as $\mathbf{y} \to \mathbf{x}$ and other terms are the same. Hence I conclude

$$\lim_{\mathbf{y}\to\mathbf{x}^+} U(\mathbf{y}) = \lim_{\mathbf{y}\to\mathbf{x}^-} U(\mathbf{y}) = U(\mathbf{x})$$

therefore U is continuous on $Q_1 \cup Q_2 \cup W^*$, that is, U being continuous on W.
Since W is compact and U is continuous, by Theorem 7, U has at least one maximiser. $\square$

13.3 Corollary 3

(a):

$$\bigcup_{k=1}^{3} V_k = \left\{ (B, I) \in \mathbb{R}^2 \middle| P_1 F_1 + B \geq I \geq 0 \text{ and } F_2(G, I) P_1 (r+1)^{-1} \geq B \right\} = W$$

(b): V_1 is convex by definition, and V_2 is convex because $F_2(NG, \cdot)$ is concave .[76]
Now observe, for an arbitrary $(B, I) \in V_1$, Equation 12a implies

$$U(B, I) = u\big(F_1 + \frac{B - I}{P_1}\big) + \beta\Big(pu\big(F_2(G, I) - \frac{B(r_d+1)}{P_1}\big) + (1-p)u\Big(F_2(NG, I) - \frac{B(r_d+1)}{P_1}\Big)\Big)$$

Since u,$F_2(G, \cdot)$, and $F_2(NG, \cdot)$ are strictly concave functions, observe that, for an arbitrary $\lambda \in (0, 1)$ and $(B', I') \in V_1$ with$(B', I') \neq (B, I)$,

[76] One may also prove by basic definition of convex sets.

$$
\begin{aligned}
U(\lambda(B,I)+(1-\lambda)(B',I')) =& u\big(F_1+\frac{\lambda(B-I)+(1-\lambda)(B'-I')}{P_1}\big) && \text{(34a)}\\
&+\beta p u\big(F_2(G,\lambda I+(1-\lambda)I')-\frac{(\lambda B+(1-\lambda)B')(r_d+1)}{P_1}\big) && \text{(34b)}\\
&+\beta(1-p)u\Big(F_2(NG,\lambda I+(1-\lambda)I')-\frac{(\lambda B+(1-\lambda)B')(r_d+1)}{P_1}\Big) && \text{(34c)}\\
>& \lambda u\big(F_1+\frac{B-I}{P_1}\big)+(1-\lambda)u\big(F_1+\frac{B'-I'}{P_1}\big) && \text{(34d)}\\
&+\beta p\lambda u\big(F_2(G,I)-\frac{B(r_d+1)}{P_1}\big) && \text{(34e)}\\
&+\beta p(1-\lambda)u\big(F_2(G,I')-\frac{B'(r_d+1)}{P_1}\big) && \text{(34f)}\\
&+\beta(1-p)\lambda u\big(F_2(NG,I)-\frac{B(r_d+1)}{P_1}\big) && \text{(34g)}\\
&+\beta(1-p)(1-\lambda)u\big(F_2(NG,I')-\frac{B'(r_d+1)}{P_1}\big) && \text{(34h)}\\
=& \lambda U(B,I)+(1-\lambda)U(B',I') && \text{(34i)}
\end{aligned}
$$

Therefore U is strictly concave on V_1. By the same process as above, U is strictly concave on V_2.

(c): Since V_1 is closed and bounded[77], $U|_{V_1}$ is continuous, by Theorem 7, there exists a maximiser, call it $a \in V_1$. Since U is strictly concave, so by Theorem 8, a is the unique maximiser, i.e. $U|_{V_1}$ has exactly one maximiser. The same argument applies to $U|_{V_2}$.

Now, notice that neither V_3 nor V_3° is convex, thus Theorem 8 cannot be used straightforward.

So, I prove by contradiction. Suppose there are more than one maximiser in V_3°. Then pick two maximisers $x, y \in V_3^\circ$ where $x \neq y$. Then by definition, $U(x) = U(y)$. Consider the line

$$l := \{\lambda x + (1-\lambda)y | \lambda \in [0,1]\}$$

$x \in V_3^\circ$ implies[78] $l^\circ \cap V_3 \neq \emptyset$, i.e. I can find $\theta \in (0,1)$ such that

$$v := \theta x + (1-\theta)y \in V_3$$

Now, observe $U(v)$, in a similar way to Equation 34 that

$$U(v) = U(\theta x + (1-\theta)y) > \theta U(x) + (1-\theta)U(y) = U(y)$$

thus y is not a maximiser, yields contradiction. Therefore $U|_{V_3^\circ}$ has at most one maximiser.

[77] This is straightforward by its definition, and W being compact.
[78] Books on metric spaces and topologies provide further details.

(d): $U|_{V_1^\circ}$ is once continuously differentiable because it is a standard composition of twice continuously differentiable functions. Moreover, $U|_{V_1}$ is concave and V_1 is convex, so by Theorem 8, x is an interior maximiser if and only if $\nabla U(x) = (0,0)$. The same argument applies to $U|_{V_2}$.

Similar to the proof in (c), V_3 is not straightforward due to it being not convex. Notice that $U|_{V_3^\circ}$ is once continuously differentiable as per argument above for $U|_{V_1^\circ}$.

$\Longrightarrow$ direction: suppose $x \in V_3^\circ$ to be an interior maximiser, then it is a local maximiser. Moreover, $U|_{V_3^\circ}$ is once continuously differentiable, so $\nabla U(x) = (0,0)$. [79]

$\Longleftarrow$ direction: suppose $x \in V_3^\circ$ satisfies $\nabla U(x) = (0,0)$. Because $x \in V_3^\circ$, there exists $\epsilon > 0$ such that ball $B := B(x, \epsilon) \subset V_3^\circ$. Now consider $U|_B$. B is convex, $U|_B$ is strictly concave[80], so by Theorem 8, x is the unique maximiser of $U|_B$.

Now (prove by contradiction) suppose x is not the interior maximiser of $U|_{V_3}$, then by definition, x is not the maximiser of $U|_{V_3}$.

Since U is continuous on V_3[81] and V_3 is compact,[82] by Theorem 7, there is one maximiser for $U|_{V_3}$, so call it $\phi \in V_3$. Notice by definition that $\phi \neq x$. Consider line

$$l := \{\lambda x + (1-\lambda)\phi | \lambda \in [0,1]\}$$

Because B is open and $x \in B$, I have $l^\circ \cap B \neq \emptyset$, so there exists $\theta \in (0,1)$ such that

$$s := \theta x + (1-\theta)\phi \in B$$

then[83]

$$U(s) = U(\theta x + (1-\theta)\phi) > \theta U(x) + (1-\theta)U(\phi) \tag{35}$$

Now, because x is the unique maximiser of $U|_B$, and $s \in B$,

$$U(s) < U(x)$$

then use Equation 35 to get $U(x) > \theta U(x) + (1-\theta)U(\phi)$, which implies

$$U(x) > U(\phi) \tag{36}$$

But Equation 36 contradicts with ϕ being the maximiser of $U|_{V_3}$. Therefore x is the interior maximiser of $U|_{V_3}$. $\square$

[79] One may refer to complex or functional analysis books for further details.

[80] One can do the same definition-fitting to Equation 34.

[81] As proved in subsection 13.2, U is continuous on W, and by definition $V_3 \subset W$.

[82] V_3 is closed by definition, and as $V_3 \subset W$ with W being bounded, I have V_3 being bounded. Therefore Theorem 7 implies V_3 is compact.

[83] In a similar way to Equation 34.

13.4 Theorem 5

$\Longrightarrow$ direction: define $S := \left\{\{B_j^*\}_{j=1}^n, A_0^*(t), A_1^*(t)\right\}$ which solves Equation 8. Consider firstly, when $t = 0$. Then Equation 21b is satisfied because $A_1^*(0)$ satisfies Equation 9b. Now suppose (to prove by contradiction) $A_0^*(0)$ does not satisfy Equation 21a, then because it satisfies Equation 9a, I have

$$A_0^*(0) > \lambda_0(L_0(0) + L_1(0)) \tag{37}$$

So, I can[84] find ξ such that

$$A_0^*(0) > \xi > \lambda_0(L_0(0) + L_1(0))$$

Then, define

$$\nu := L_0(0) + L_1(0) - \xi - \sum_{j=1}^{n} B_j^*$$

Now, by Lemma 4, the combination $\left\{\{B_j^*\}_{j=1}^n, \xi, \nu\right\}$ yields[85] higher expected utility than S, thus contradicts with S solving Equation 8. Hence $A_0^*(t)$ satisfies Equation 21a. Now since S maximises Equation 8 which has more choice variables compared to Equation 20, and Equation 21a & Equation 21b being satisfied by S, I conclude S solves Equation 20 when $t = 0$.

Now consider when $t = 1$. Equation 21c is satisfied because $A_0^*(1)$ satisfies Equation 9a. Using the same argument as above, I can prove by contradiction that, Equation 21d must be satisfied because of Lemma 4. Therefore, as argued above, because S maximises Equation 8 and Equation 21c & Equation 21d being satisfied, I conclude S solves Equation 20 when $t = 1$.

$\Longleftarrow$ direction: define $S := \left\{\{B_j^*\}_{j=1}^n, A_0^*(t), A_1^*(t)\right\}$ which solves Equation 20. Notice that, for both $t = 0$ and $t = 1$, as Equation 21 is satisfied, Equation 9 is automatically satisfied. Now suppose (to prove by contradiction) S does not solve Equation 20, then there exists another set $\widehat{S} := \left\{\{\beta_j^*\}_{j=1}^n, \alpha_0^*(t), \alpha_1^*(t)\right\}$ that solves Equation 8, and $S \neq \widehat{S}$. This means $\widehat{S}$ yields higher expected utility than S. I state, then prove the following two claims

Claim One:[86] $\{\beta_j^*\}_{j=1}^n = \{B_j^*\}_{j=1}^n$
Claim Two: $\alpha_0^*(t) = A_0^*(t)$

[84] Because $\mathbb{R}$ is complete and dense in itself. Or more practically, whenever $a, b \in \mathbb{R}$ and $a > b$, there always exists $\varepsilon > 0$ such that $a > a - \varepsilon > b$.
[85] Note: this combination still satisfies Equation 9.
[86] The "=" here shall be seen as equal up to bijective permutation, e.g. under this setting, $\{1, 2, 2\} = \{2, 1, 2\} = \{2, 2, 1\}$

Prove Claim One by contradiction: suppose $\{\beta_j^*\}_{j=1}^n \neq \{B_j^*\}_{j=1}^n$, then define

$$\theta = L_0(0) + L_1(0) - A_0^*(0) - \sum_{j=1}^n \beta_j^*$$
$$\delta = L_0(1) + L_1(1) - A_1^*(1) - \sum_{j=1}^n \beta_j^*$$

Now observe, $Y := \{\{\beta_j^*\}_{j=1}^n, A_0^*(0), \theta\}$ and $Z := \{\{\beta_j^*\}_{j=1}^n, \delta, A_1^*(1)\}$ satisfy[87] Equation 21 and $Y \neq S$, $Z \neq S$. Then, by Lemma 4, Y yields at least as much expected utility as $\widehat{S}$, and so do Z. But $\widehat{S}$ yields higher expected utility than S, so Y and Z solves Equation 20, and has higher expected utility than S, which contradicts with S solving Equation 20.

Prove Claim Two by contradiction: suppose $\alpha_0^*(t) \neq A_0^*(t)$, then I reach contradiction for both $t = 0$ and $t = 1$ as shown below.

When $t = 0$, $A_0^*(0)$ satisfies Equation 21a and $\alpha_0^*(0)$ satisfies Equation 9a, so $\alpha_0^*(0) \neq A_0^*(0)$ implies

$$\alpha_0^*(0) > \lambda_0(L_0(0) + L_1(0))$$

Then, as the same argument stated after Equation 37, I can find another combination that yields higher expected utility than $\widehat{S}$, thus contradicts with $\widehat{S}$ solving Equation 8.

When $t = 1$, because $\{\beta_j^*\}_{j=1}^n = \{B_j^*\}_{j=1}^n$ by Claim One and $\alpha_0^*(1) \neq A_0^*(1)$, I get

$$\begin{aligned}\alpha_1^*(1) = L_0(1) + L_1(1) - \sum_{j=1}^n \beta_j^* - \alpha_0^*(1) =& L_0(1) + L_1(1) - \sum_{j=1}^n B_j^* - \alpha_0^*(1) \\ \neq& L_0(1) + L_1(1) - \sum_{j=1}^n B_j^* - A_0^*(1) = A_1^*(1)\end{aligned}$$

Since $A_1^*(1)$ satisfies Equation 21d and $\alpha_1^*(1)$ satisfies Equation 9b, I have $\alpha_1^*(1) > \lambda_1(L_0(1) + L_1(1))$. Then, as the same argument stated after Equation 37, I can find another combination that yields higher expected utility than $\widehat{S}$, thus contradicts with $\widehat{S}$ solving Equation 8.

Now, Claim One and Two are proved, thus

$$\alpha_1^*(t) = L_0(t) + L_1(t) - \sum_{j=1}^n \beta_j^* - \alpha_0^*(t) = L_0(t) + L_1(t) - \sum_{j=1}^n B_j^* - A_0^*(t) = A_1^*(t)$$

So $\left\{\{\beta_j^*\}_{j=1}^n, \alpha_0^*(t), \alpha_1^*(t)\right\} = \left\{\{B_j^*\}_{j=1}^n, A_0^*(t), A_1^*(t)\right\}$, which contradicts with $S \neq \widehat{S}$.

Therefore S solves Equation 20. $\square$

[87] This is because $\widehat{S}$ satisfies Equation 9, then by construction, the other two equalities are satisfied.

13.5 Theorem 6

Consider the following two sets of functions:

$$\mathscr{A} = \left\{u : [-2, +\infty) \to \mathbb{R}, \quad x \mapsto k + f(x) \middle| k \in \mathbb{R}\right\}$$
$$\mathscr{B} = \left\{u : [-2, +\infty) \to \mathbb{R}, \quad x \mapsto k + \log(x + 3) \middle| k \in \mathbb{R}\right\}$$

where $f : [-2, +\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \log(x + 0.170901) & \text{if} \quad x \geq -0.1709 \\ \rho_3 x^3 + \rho_2 x^2 + \rho_1 x + \rho_0 & \text{otherwise} \end{cases}$$

where $\{\rho_j\}_{j=0}^{3}$ are such that f is twice continuously differentiable over the ball centred at -0.1709. [88]

Thus, by construction, every member in $\mathscr{A} \cup \mathscr{B}$ satisfies Setting 11.

Pick an arbitrary $u \in \mathscr{A}$, then observe

$$u(0.0160) = k + f(0.016) < k - 1.67 < 0.99u(0.0247) + 0.01u(-0.1596)$$
$$u(0.4i_c) = k + f(-0.004) > k - 1.8 > 0.99u(0.0173 + 0.4i_c) + 0.01u(-0.1669 + 0.4i_c)$$

So, every member of $\mathscr{A}$ satisfies Equation 25a and Equation 25c. Since $\mathscr{A}$ is uncountable, so there are uncountably many utility functions that make $ComB$ to grant credit when $i_c = 0.01$, but not when $i_c = -0.01$.

Similarly, pick an arbitrary $u \in \mathscr{B}$, then observe

$$u(0.0160) = k + \log(3.16) < k + 1.1 < 0.99u(0.0247) + 0.01u(-0.1596)$$
$$u(0.4i_c) = k + \log(2.996) < k + 1.1 < 0.99u(0.0173 + 0.4i_c) + 0.01u(-0.1669 + 0.4i_c)$$

So, every member of $\mathscr{B}$ satisfies Equation 25a and Equation 25b. Since $\mathscr{B}$ is uncountable, so there are uncountably many utility functions that make $ComB$ to grant credit both when $i_c = 0.01$, and when $i_c = -0.01$. □

[88] More rigorously, $\{\rho_j\}_{j=0}^{3}$ is the unique solution to the system

$$\begin{aligned} \rho_3(-0.1709)^3 + \rho_2(-0.1709)^2 + \rho_1(-0.1709) + \rho_0 &= \log(0.000001) \\ 3\rho_3(-0.1709)^2 + 2\rho_2(-0.1709) + \rho_1 &= (0.000001)^{-1} \\ 6\rho_3(-0.1709) + 2\rho_2 &= -(0.000001)^{-2} \\ 3\rho_3 &= (0.000001)^{-3} \end{aligned}$$

Part V

References

Note: Matlab codes are available upon reasonable request.

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